Solution 11.

11.15 (a) $$\xi_j(x) = \int G_{jk}(x-x_0) F_k \delta^2(x_0) d^2 x_0$$
$$= G_{jk}(x) F_k$$

(b) $$\begin{cases} \xi_z = G_{zz}(\varpi, z) F_z = \dfrac{(1+\nu)}{2\pi E}\left[\dfrac{2(1-\nu)}{\varpi} + \dfrac{z^2}{\varpi^3}\right] F_z \\ \xi_\varpi = G_{\varpi z}(\varpi, z) F_z = -\dfrac{(1+\nu)}{2\pi E}\left[\dfrac{1-2\nu}{\varpi+z} - \dfrac{z}{\varpi^2}\right] F_z \end{cases}$$

$$\Sigma_{z\varpi} = \frac{1}{2}\frac{\partial \xi_\varpi}{\partial z} + \frac{1}{2}\frac{\partial \xi_z}{\partial \varpi} = \frac{1}{2}\left(\frac{1+\nu}{2\pi E}\left(\frac{1}{\varpi^2} + \frac{1-2\nu}{(\varpi+z)^2}\right) + \frac{1+\nu}{2\pi E}\left(\frac{-2(1-\nu)}{\varpi^2} - \frac{3z^2}{\varpi^4}\right)\right) F_z$$
$$= 0 \quad \Rightarrow T_{z\varpi} = 0$$

It's straight forward to check that the elastic equations are satisfied.

(c) $$T_{zz} = -K\Theta - 2\mu \Sigma_{zz}$$

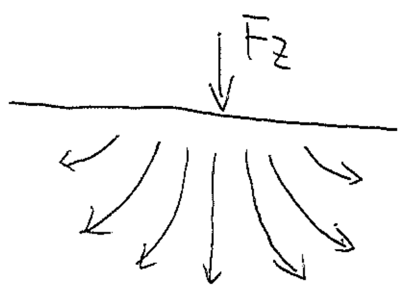
$$\Theta = \frac{1+\nu}{2\pi E} F_z \left(\frac{1-2\nu}{(\varpi+z)^2}\frac{z}{\varpi} + \frac{z}{\varpi^3}\right)$$

$$\Sigma_{zz} = \frac{1}{3}\frac{1+\nu}{2\pi E} F_z \left(\frac{5z}{\varpi^3} + \frac{(1-2\nu)}{(\varpi+z)^2}\frac{z}{\varpi}\right)$$

$$\Rightarrow \begin{cases} T_{zz}|_{z=0,\ \varpi\neq 0} = 0 \\ T_{zz}|_{z=0,\ \varpi=0} = \infty \end{cases}$$ Moreover $$\int T_{zz(z=0)}\, d\Sigma = F_z$$

$$\Rightarrow T_{zz}(z=0) = F_z \delta_2(\vec{x})$$

(d) The vector displacement $\vec{\xi}$ look like



The pattern is axis-symmetric.

11.17 (a) Consider the part of the wire between one end and the point a distance z' from the end. e_z component of the total force

$$\begin{cases} F\cos\theta = \int T_{z'z'}\,dx'dy' \\ F\sin\theta = \int T_{x'z'}\,dx'dy' \end{cases}$$

(b) Torque balance:

$$F\sin\theta = S = \frac{M(z'+dz') - M(z')}{dz'} = \frac{dM}{dz'}$$

with $M = \int x' T_{z'z'}\,dx'dy'$

(c) By using $T_{z'z'} = -E\,S_{z'z'} = -Ex'\frac{d\theta}{dz'}$ and performing the integral in part b $\Rightarrow$ $M = -D\frac{d\theta}{dz'}$ with $D = Ewh^3/12$

Solution 11

(d) Combining the above results we immediately have

$$\frac{d^2\theta}{d\underset{\sim}{x}^2} = -\frac{F\sin\theta}{D}$$

(f) Mathematica gives a nontrivial solution

$$\theta(\underset{\sim}{x}) = 2\,\mathrm{am}\left(\frac{1}{\sqrt{2}}\frac{\underset{\sim}{x}}{\ell}, 2\right)$$

where $\mathrm{am}(u, m)$ is the inverse function of the elliptic integral of the first kind: $F(\phi, m) = \int_0^\phi \frac{1}{\sqrt{1-m\sin^2 t}}\,dt$

(g) $\underset{\sim}{x} = \sqrt{2}\,\ell \int_0^{\theta/2} \frac{1}{\sqrt{1-2\sin^2 t}}\,dt$

$$\Rightarrow d\underset{\sim}{x} = \frac{\ell}{\sqrt{2}}\frac{1}{\sqrt{\cos\theta}}\,d\theta$$

$$\Rightarrow \begin{cases} \cos\theta = \dfrac{dx}{d\underset{\sim}{x}} = \sqrt{2\cos\theta}\,\dfrac{1}{\ell}\dfrac{dx}{d\theta} \\ \sin\theta = \dfrac{dz}{d\underset{\sim}{x}} = \sqrt{2\cos\theta}\,\dfrac{1}{\ell}\dfrac{dz}{d\theta} \end{cases} \Rightarrow \cos\theta = \left(\frac{z}{\sqrt{2}}-1\right)^2$$

$$\Rightarrow \frac{dx}{dz} = \frac{\cos\theta}{\sin\theta} = \frac{\left(\frac{z}{\sqrt{2}}-1\right)^2}{\pm\sqrt{1-\left(\frac{z}{\sqrt{2}}-1\right)^4}}$$

Solution 11

12.8: Free-energy analysis of buckling instability [by Dan Grin]
(a) From Eq. (10.23) in the course notes, we know that the total elastic energy is

$$\mathcal{E} = \int \frac{1}{2} E \left(\xi_{x,x}\right)^2 dxdydz. \tag{1}$$

From Eq. (10.77) in the notes we know that $\xi_{x,x} \simeq z\frac{d^2\eta}{dx^2}$ and from Eq. (10.82) in the notes, we know that $D \equiv \int z^2 dydz$, so we have

$$\boxed{\mathcal{E} = \tfrac{1}{2} D \int_0^l \left(\tfrac{\partial^2 \eta}{\partial x^2}\right)^2 dx.}$$

(b) Looking at Fig. 10.10 in the class notes, we see for a differential deflection $d\eta$ and a length change along the neutral surface dx, the horizontal length of the beam changes by $dx - \sqrt{dx^2 - d\eta^2}$ (this is the implied application of the Pythagorean theorem). At lowest order, this means $d\delta l = -\frac{1}{2}\left(\frac{d\eta}{dx}\right)^2 dx$, and so $\delta l = -\int_0^l \left(\frac{d\eta}{dx}\right)^2 dx$, and so the total free energy is

$$\boxed{H = \tfrac{1}{2}\int_0^l \left[D\left(\tfrac{\partial^2\eta}{\partial x^2}\right)^2 - F\left(\tfrac{d\eta}{dx}\right)^2\right] dx.}$$

(c) Since pressure P is just force F per unit area, $F = PA = Phw$, and so the second term is $-F\delta l = -Phw\delta l = -P\delta V$. Thus the free energy is just the enthalpy:

$$\boxed{H = \mathcal{E} + P\delta V.}$$

If we apply the variation $\delta\eta(x)$, demanding that $\delta\eta(0) = \delta\eta(l) = 0$ and repeatedly switching variational derivatives with partials, integrating by parts, and throwing away surface terms when $\delta\eta = 0$ permits, we obtain

$$\left[D\frac{d^2\eta}{dx^2}\delta\left(\frac{d\eta}{dx}\right)\right]\Bigg|_{x=0}^{x=l} + D\nabla^4\eta + F\nabla^2\eta = 0. \tag{2}$$

Symmetry about the nodes tells us that $d\eta/dx(x = 0^+) = d\eta/dx(x = l^-)$. If we imagine doubling the length of our rod, and realize that for the same applied force, the solution must repeat itself, we realize that we also have translational symmetry at our disposal. In other words, $d\eta/dx(x = 0^+) = d\eta/dx(x = l^+)$, and so $d\eta/dx(x = l^-) = d\eta/dx(x = l^+)$. In other words, $d^2\eta/dx^2 = 0$, and the ends of the rod are inflection points. Thus

$$\boxed{D\nabla^4\eta + F\nabla^2\eta = 0,}$$

as desired. Since this just Eq. (11.39) in the online notes with $\ddot{\eta} = 0$, the spatial eigen-functions are

$$\boxed{\eta = \eta_i \sin \tfrac{n\pi x}{l}.}$$

(d) For the $n = 0$ case, all derivatives are 0, so we need only calculate the $n = 1$ case:

$$\left(\nabla^2\eta\right)^2 = \frac{\pi^4}{l^4}\eta_0^2 \sin^2\left(\frac{\pi x}{l}\right) \tag{3}$$

$$\left(\nabla\eta\right)^2 = \frac{\pi^2}{l^2}\eta_0^2 \cos^2\left(\frac{\pi x}{l}\right). \tag{4}$$

Performing the substitution $z = \pi x/l$ and recalling that $\sin^2 x = \frac{1-\cos 2x}{2}$ and $\cos^2 x = \frac{1+\cos 2x}{2}$, we see that Eq. (4) yields the desired result:

$$\boxed{H_1 - H_0 = \left(\tfrac{\pi\eta_0}{2l}\right)^2 l(F_{\text{crit}} - F)}$$

$$\boxed{F_{\text{crit}} = \tfrac{\pi^2 D}{l^2}}$$

.

12.4: Lagrangian and energy density for elastodynamic waves [by Geoffrey Lovelace]
(a) Begin with the assumed form of the stress tensor,

$$T_{ij} = -Y_{ijkl}\xi_{k;l} = -Y_{ijkl}S_{kl}. \tag{5}$$

If the medium is isotropic, Y can be written as a linear combination of constant tensors. Also, we know that T_{ij} must be symmetric. The only rank-two, symmetric, constant tensor that we have is the metric $g_{ij} = \delta_{ij}$.

Now, Y_{ijkl} is symmetric in i and j, symmetric in k and l, and symmetric under interchange of the pairs ij and kl. It must also be constructed from Kronecker deltas. The most general tensor you can write that has the required symmetries is

$$Y_{ijkl} = Ag_{ij}g_{kl} + B(g_{ik}g_{jl} + g_{il}g_{jk}), \tag{6}$$

where A and B are scalars.

To evaluate A and B, compare $T_{ij} = Y_{ijkl}\xi_{k;l}$ to Eq. (10.38), which defines the bulk modulus and the shear modulus. Contracting Y_{ijkl} into the strain gives

$$T_{ij} = -\left(A + \frac{2}{3}B\right)\theta\delta_{ij} - 2B\Sigma_{ij}. \tag{7}$$

while Eq. (10.38) says

$$T_{ij} = -Y_{ijkl}S_{kl} = -K\theta\delta_{ij} - 2\mu\Sigma_{ij}. \tag{8}$$

Therefore, $B = \mu$ and $K = A + (2/3)B \Rightarrow A = K - (2/3)\mu$.
(b) This is a basic exercise in the calculus of variations. For this problem, I find it more convenient to use spacetime indices for a while. The action for a Lagrange density $\mathcal{L}$ is

$$S = \int d^4x \mathcal{L}(\xi_\mu, \partial_\nu \xi_\mu). \tag{9}$$

Here, $d^4x = dtd^3x$, μ and ν are spacetime indices. Note that I choose to work in coordinates such that $g_{00} = -1$ and $g_{0j} = 0$. This implies that you can pass between semicolons and commas when the differentiation index is a temporal index: $a_{,0} = a_{;0} = \dot{a}$.

The equation of motion is found by extremizing the action:

$$\delta S = 0 = \int d^4x \delta \left[\mathcal{L}(\xi_\mu, \xi_{\mu;\nu})\right]. \tag{10}$$

The variation of the Lagrange density is

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\xi_\mu}\delta\xi_\mu + \frac{\partial\mathcal{L}}{\partial\xi_{\mu;\nu}}\delta(\xi_{\mu;v}) = 0. \tag{11}$$

Integrating by parts and throwing away the surface term[1] in the second term implies that

$$\delta\mathcal{L} = 0 = \frac{\partial\mathcal{L}}{\partial\xi_\mu}\delta\xi_\mu - \nabla_\nu\left(\frac{\partial\mathcal{L}}{\partial\xi_{\mu;\nu}}\right)\delta\xi_\mu. \tag{12}$$

Since $\delta\xi_\mu$ is arbitrary, the equation of motion is

$$\delta\mathcal{L} = 0 = \frac{\partial\mathcal{L}}{\partial\xi_\mu} - \nabla_\nu\left(\frac{\partial\mathcal{L}}{\partial\xi_{\mu;\nu}}\right). \tag{13}$$

Our particular Lagrange density only depends on derivatives of ξ, and ξ is a spatial vector (in the preferred frame we are working in). This makes it is easy to go back to a notation where time and space are separated, now that we have done the variational calculation. The equation of motion is then

$$\delta\mathcal{L} = 0 = \nabla_0\left(\frac{\partial\mathcal{L}}{\partial\xi_{i;0}}\right)\delta\xi_i + \nabla_j\left(\frac{\partial\mathcal{L}}{\partial\xi_{i;j}}\delta\xi_i\right) \tag{14}$$

$$\Rightarrow 0 = \frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\xi}_i}\right) + \nabla_j\left(\frac{\partial\mathcal{L}}{\partial\xi_{i;j}}\right) \tag{15}$$

[1] As usual in this sort of calculation, I assume that the surface term vanishes, since I suppose the integration is over all spacetime.

For the Lagrange density we are given,

$$\mathcal{L} = \frac{1}{2}\rho\dot{\xi}_i\dot{\xi}_i - \frac{1}{2}Y_{ijkl}\xi_{i;j}\xi_{k;l}, \tag{16}$$

the equation of motion is

$$\rho\ddot{\xi}_i = (Y_{ijkl}\xi_{k;l})_{;j} = -T_{ij;j}. \tag{17}$$

This is an analog of $F = ma$.
(c) Verifying the conservation law is straightforward.

$$\begin{aligned} U &= \frac{1}{2}\rho\dot{\xi}_i\dot{\xi}_i + \frac{1}{2}Y_{ijkl}\xi_{i;j}\xi_{k;l} \\ \Rightarrow \frac{\partial U}{\partial t} &= \rho\dot{\xi}_i\ddot{\xi}_i + Y_{ijkl}\dot{\xi}_{i;j}\xi_{k;l} = -\dot{\xi}_i T_{ij;j} - \dot{\xi}_{i;j}T_{ij} \\ &= -\nabla_j\left(\dot{\xi}_i T_{ij}\right) = -\nabla_j F_j. \end{aligned} \tag{18}$$

Therefore,

$$\frac{\partial U}{\partial t} + \nabla\cdot\mathbf{F} = 0. \tag{19}$$

(d) The idea in this part is to recognize that we are dealing with a monochromatic, plane wave mode. Therefore, derivatives can be manipulated algebraically. For instance, the kinetic energy density is

$$\begin{aligned} \left\langle U^{KE}\right\rangle &= \frac{1}{2}\rho\left\langle\dot{\xi}_i\dot{\xi}_i\right\rangle = \frac{1}{2}\rho\left\langle(-i\omega)\xi_i(-i\omega)\xi_i\right\rangle = \frac{1}{2}\rho\left\langle(-i\omega)^2\xi_i\xi_i\right\rangle \\ &= \frac{1}{2}\left\langle\rho\ddot{\xi}_i\xi_i\right\rangle = -\frac{1}{2}\left\langle T_{ij;j}\xi_i\right\rangle \end{aligned} \tag{20}$$

Meanwhile, if $e_{\mathbf{k}j}$ is the jth component of the unit vector pointing in the $\mathbf{k}$ direction,

$$\begin{aligned} \left\langle U^{el}\right\rangle &= \frac{1}{2}\left\langle Y_{ijkl}\xi_{i;j}\xi_{k;l}\right\rangle = \frac{1}{2}\left\langle Y_{ijkl}(ik)\xi_i(ik)\xi_k e_{\mathbf{k}j}e_{\mathbf{k}l}\right\rangle \\ &= \frac{1}{2}\left\langle Y_{ijkl}\xi_i(ik)^2 e_{\mathbf{k}j}e_{\mathbf{k}l}\xi_k\right\rangle = \frac{1}{2}\left\langle\xi_i(Y_{ijkl}\xi_{k;l})_{;j}\right\rangle \\ &= -\frac{1}{2}\left\langle T_{ij;j}\xi_i\right\rangle \Rightarrow \left\langle U^{KE}\right\rangle = \left\langle U^{el}\right\rangle. \end{aligned} \tag{21}$$

Since both contributions to the energy density are equal,

$$\langle U\rangle = \rho\left\langle\dot{\xi}_i\dot{\xi}_i\right\rangle. \tag{22}$$

Now, let's get the energy densities for transverse and longitudinal modes. For the longitudinal case, $\xi_i = \xi e_{\mathbf{k}i}$, so

$$\langle U_L\rangle = \rho\left\langle\dot{\xi}_i\dot{\xi}_i\right\rangle = \rho\left\langle\dot{\xi}^2 e_{\mathbf{k}i}e_{\mathbf{k}i}\right\rangle = \rho\left\langle\dot{\xi}^2\right\rangle. \tag{23}$$

Similarly, for transverse modes, $\xi_i = \xi_i^T$ is orthogonal to k_i, and

$$\langle U_T \rangle = \rho \left\langle (-i\omega)^2 \xi_i^T \xi_i^T \right\rangle = -\rho\omega^2 \left\langle \xi^T \cdot \xi^T \right\rangle . \tag{24}$$

Note: I get a different sign than Eq. (11.15c). My sign is correct because each of the two time derivatives must bring down a factor of $-i\omega$, and $(-i\omega)^2 = -\omega^2$. Otherwise, this recovers the energy densities given in the text.

(e) Finally, we want to verify Eq. (11.15), which tells us the longitudinal and transverse fluxes for isotropic media. So the first thing to do is to specialize F_i, given in Eq. (11.21), to an isotropic medium. This amounts to nothing more than inserting Eq. (11.18) for Y_{ijkl} into Eq. (11.21). Since the metric is just $g_{ij} = \delta_{ij}$, it is trivial to show that

$$\begin{aligned}
F_i &= -(K - \frac{2}{3}\mu)\dot{\xi}_j \xi_{k;k} - \mu \left(\dot{\xi}_k \xi_{k;j} + \dot{\xi}_l \xi_{j;l} \right) \\
&= -(K - \frac{2}{3}\mu)\dot{\xi}_j \xi_{k;k} - \mu \left(\dot{\xi}_k \xi_{k;j} + (-i\omega)\xi_l e_{\mathbf{k}l}(ik)\xi_j \right) \\
&= -(K - \frac{2}{3}\mu)\dot{\xi}_j \xi_{k;k} - \mu \left(\dot{\xi}_k \xi_{k;j} + e_{\mathbf{k}l}(ik)\xi_l(-i\omega)\xi_j \right) \\
&= -(K - \frac{2}{3}\mu)\dot{\xi}_j \theta - \mu \left(\dot{\xi}_k \xi_{k;j} + \theta\dot{\xi}_j \right) \\
&= -(K + \frac{1}{3}\mu)\dot{\xi}_j \theta - \mu \left(\dot{\xi}_k \xi_{k;j} \right)
\end{aligned} \tag{25}$$

Notice that I have a factor of $\frac{1}{3}$, not $\frac{2}{3}$ as written in the text.

Next, treat longitudinal waves. In this case, $\xi = \xi e_{\mathbf{k}}$.

$$\begin{aligned}
\langle F_j \rangle_L &= -\left[(K + \frac{1}{3}\mu) \left\langle \dot{\xi} e_{\mathbf{k}j} \xi (ik) e_{\mathbf{k}l} e_{\mathbf{k}l} \right\rangle + \mu \left\langle \dot{\xi}\xi \right\rangle e_{\mathbf{k}j}(ik) \right] \\
&= -\left[-\frac{(K + \frac{4}{3}\mu)}{c_L} \left\langle \dot{\xi}^2 e_{\mathbf{k}j} \right\rangle \right] = c_L \rho e_{\mathbf{k}j} \left\langle \dot{\xi}^2 \right\rangle = c_L U_L e_{\mathbf{k}j}
\end{aligned} \tag{26}$$

For transverse waves, the first term vanishes, because $\theta = \xi_{i;i} = ik_i\xi_i = 0$, since the displacement is orthogonal to the wavevector. The remaining term gives

$$\langle F_j \rangle_T = -\mu \left\langle \xi_i \xi_i e_{\mathbf{k}j} \frac{\omega^2}{c_T} \right\rangle = c_T U_T e_{\mathbf{k}j} \tag{27}$$

Solution 12

2. (b) $\vec{\nabla}\cdot(f\vec{Y}) = \vec{\nabla}\cdot(f\vec{e}_r Y) = \frac{\partial}{\partial r}(f)Y + (\vec{\nabla}\cdot\vec{e}_r)fY$ ①

$= (f' + 2f/r)Y$

$\vec{\nabla}\cdot(f\vec{\Psi}) = \vec{\nabla}\cdot(fr\vec{\nabla}Y) = \underbrace{\nabla(fr)\cdot\nabla Y}_{=0} + fr\nabla^2 Y$ ②

$= -\frac{\ell(\ell+1)}{r}fY$

$\vec{\nabla}\cdot(f\vec{\Phi}) = \vec{\nabla}\cdot(f\vec{r}\times\nabla Y) = 0$ ③

It's easy to check

$\vec{\nabla}\cdot(\omega\vec{\Phi}) = 0$ (③)

$\vec{\nabla}\cdot\left(\ell(\ell+1)\frac{u}{r}\vec{Y} + (\partial_r u + \frac{u}{r})\vec{\Psi}\right) = \ell(\ell+1)\left[(\frac{u}{r})' + \frac{2u}{r^2}\right]Y - \frac{\ell(\ell+1)}{r}(\partial_r u + \frac{u}{r})Y$

$= 0$

So they can represent transverse mode.

(c) $\vec{\nabla}\times(f\vec{Y}) = (\vec{\nabla}\times\vec{e}_r)fY + \vec{\nabla}(fY)\times\vec{e}_r = f(\nabla Y)\times\vec{e}_r$

$= -\frac{f}{r}\vec{\Phi}$

$\vec{\nabla}\times(f\vec{\Psi}) = (f' + \frac{f}{r})\vec{r}\times\vec{\nabla}Y + fr\vec{\nabla}\times\vec{\nabla}Y = (f' + \frac{f}{r})\vec{\Phi}$

$\vec{\nabla}\times(f\vec{\Phi}) = \vec{\nabla}\times[f(\vec{r}\times\nabla Y)] = (f\vec{r})\nabla^2 Y - \nabla Y(\vec{\nabla}\cdot(f\vec{r}))$

$= -\frac{\ell(\ell+1)}{r}f\vec{Y} - (f' + \frac{f}{r})\vec{\Psi}$

Solution 12

(c) It's straight forward to check that

$$\partial_r z \vec{Y} + \frac{z}{r}\vec{\Psi} \quad \text{Satisfies} \quad \vec{\nabla}\times(\partial_r z\vec{Y} + \frac{z}{r}\vec{\Psi}) = 0$$

(d) $\vec{\nabla}\times(\nabla\times\vec{\xi}) \Rightarrow \epsilon_{ijk}(\epsilon_{k\ell m}\xi_{m;\ell})_{;j} = \epsilon_{ijk}\epsilon_{k\ell m}\xi_{m;\ell j}$

$= (\delta_{i\ell}\delta_{jm} - \delta_{im}\delta_{j\ell})\xi_{m;\ell j} = \xi_{j;ji} - \xi_{i;jj}$

$= \vec{\nabla}(\vec{\nabla}\cdot\vec{\xi}) - \nabla^2\vec{\xi}$

3. (a) $\rho\ddot{\vec{\xi}} = (K + \frac{\mu}{3})\nabla(\nabla\cdot\vec{\xi}) + \mu\nabla^2\vec{\xi}$

for $\vec{\xi} = \bar{\omega}\vec{\Phi}$, $\nabla\cdot\vec{\xi} = 0$, $\Rightarrow \rho\ddot{\vec{\xi}} = \mu\nabla^2\vec{\xi}$

or $\quad \dfrac{\omega^2}{C_T^2}\bar{\omega} + \dfrac{1}{r^2}\dfrac{\partial}{\partial r}(r^2\dfrac{\partial\bar{\omega}}{\partial r}) - \dfrac{\ell(\ell+1)}{r^2}\bar{\omega} = 0 \qquad C_T^2 = \dfrac{\mu}{\rho}$

Apprently spherical Bessel function $j_\ell(k_T r)$ & $n_\ell(k_T r)$ are solutions to this equation. In order to be regular at $r=0$ ($\bar{\omega}=0$ @ $r=0$), only $j_\ell(k_T r)$ is allowed. and the starting ℓ is 1.

(b) for $\vec{\xi} = \partial_r z\vec{Y} + \frac{z}{r}\vec{\Psi}$, it's straight forward to show $\dfrac{\omega^2}{C_L^2}z + \dfrac{1}{r^2}\dfrac{\partial}{\partial r}(r^2\dfrac{\partial z}{\partial r}) - \dfrac{\ell(\ell+1)}{r^2}z = 0$, $\quad C_L^2 = \dfrac{K + \frac{4}{3}\mu}{\rho}$