FUNDAMENTAL LIMITS ON THERMODYNAMIC COSTS OF CIRCUITS

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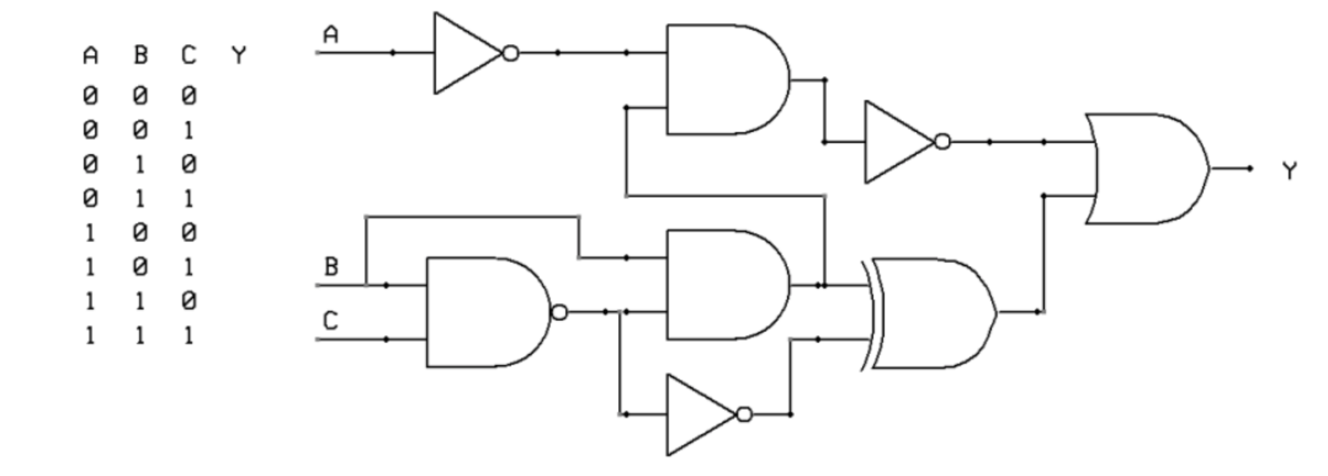
- ~5% energy use in developed countries goes to heating digital circuits
- Cells “compute” ~ 10^5 times more efficiently than CMOS computers
- In fact, *biosphere as a whole* is more efficient than supercomputers
- Deep connections among information theory, physics and (neuro)biology

Analyze relation between circuit's design and the heat it produces

- Calculate heat produced by running each gate in a circuit
- Sum over all gates to get total heat produced by running the full circuit



- Must be able to calculate heat produced by running an arbitrary gate…

(*See* N. Gershenfeld, IBM Systems Journal 35, 577 (1996))

Consider a (perhaps time-varying) master equation that sends $p_0(x)$ to $p_1(x) = \sum_{x_0} P(x_1 \mid x_0)\, p_0(x)$.

- Example: Stochastic dynamics in a genetic network
- Example: (Noise-free) dynamics of a digital gate in a circuit

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- Example: Stochastic dynamics in a genetic network network

- Example: (Noise-free) dynamics of a digital gate in a circuit

$$S(p_1) - S(p_0) = EF(p_0) + EP(p_0)$$

where:

- $S(p)$ is Shannon entropy of p

- $EF(p_0)$ is total entropy flow (into system) between $t = 0$ and $t = 1$

- $EP(p_0)$ is total entropy production in system between $t = 0$ and $t = 1$

(*For the moment, precise definitions of EF and EP omitted*)

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- Example: Stochastic dynamics in a genetic network network
- Example: (Noise-free) dynamics of a digital gate in a circuit

$$EF(p_0) = S(p_0) - S(p_1) + EP(p_0)$$

$EP(p_0)$ is non-negative (regardless of the master equation)

(See Van Den Broeck and Esposito, *Physica A*, 2015)

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$$EF(p_0) \geq S(p_0) - S(p_1)$$

"Generalized Landauer's bound"

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"Generalized Landauer's bound"

So far, all math, no physics …

STATISTICAL PHYSICS APPLICATION

Suppose system evolves while connected to multiple reservoirs, e.g., heat baths at different temperatures.

Assume "local detailed balance" holds for those reservoirs (usually true)

Then: ***EF(p_0) is (temperature-normalized) heat flow into reservoirs***

$$EF(p_0) = S(p_0) - S(p_1) + EP(p_0)$$

Generalized Landauer's bound:
Heat flow from system $\geq S(p_0) - S(p_1)$

$S(p_0) - S(p_1)$ called "Landauer cost" – minimal possible heat flow

EXAMPLE

- System evolves while connected to *single* heat bath at temperature T
- Two possible states
- p_0 uniform
- Process implements bit erasure (so p_1 a delta function)

So generalized Landauer's bound says

$$\text{Total heat flow from system} \geq kT \ln[2]$$

Landauer's conclusion

(*Parrondo et al. 2015, Sagawa 2014, Hasegawa et al. 2010, Wolpert 2015, etc.*)

BACK TO EXACT RESULTS

$$EF(p_0) = S(p_0) - S(p_1) + EP(p_0)$$

- For fixed $P(x_1 \mid x_0)$, changing p_0 changes $S(p_0) - S(p_1)$
- Same physical gate (e.g., an AND gate, made in a particular factory) has different p_0, depending on where it is in a circuit.
- So identical gates at different locations in a circuit have different Landauer costs, $S(p_0) - S(p_1)$

Different circuits all implementing same Boolean function have different sum-total Landauer cost

➢ A new circuit design optimization problem

- But EF is the Landauer cost *plus the EP*:

$$EF(p_0) = S(p_0) - S(p_1) + EP(p_0)$$

- For fixed $P(x_1 \mid x_0)$, e.g., a fixed gate, changing p_0 changes $EP(p_0)$ as well as changing $S(p_0) - S(p_1)$

- So identical gates at different locations in a circuit will have different EP

Need to know how $EP(p_0)$ depends on p_0

… just to define the optimization problem
of designing a circuit to minimize total EF
- never mind solve that problem

Theorem: For any p_0, and any master equation, entropy production is

$$EP(p_0) = KL\big[p_0(X), q_0(X)\big] - KL\big[p_1(X), q_1(X)\big] + \sum_c p_0(c) EP(q_0^c)$$

where:

- KL(., .) is Kullback-Leibler divergence
- $q_0(x)$ is a “prior” built into the system; EP is minimal if $p_0 = q_0$ and EP is nonzero if they differ, i.e., if you “guess wrong” when designing the physical system.
- The sum is over “islands” (graph theory) of the dynamics.

(*Kolchinsky and Wolpert 2017, Kolchinsky and Wolpert 2018*)

So entropy flow out of a system – the thermodynamic cost – is

$$EF(p_0) = K\big[p_0(X), q_0(X)\big] - K\big[p_1(X), q_1(X)\big] + \sum_c p_0(c) EP(q_0^c)$$

where K(., .) is cross–entropy.

Applies to:

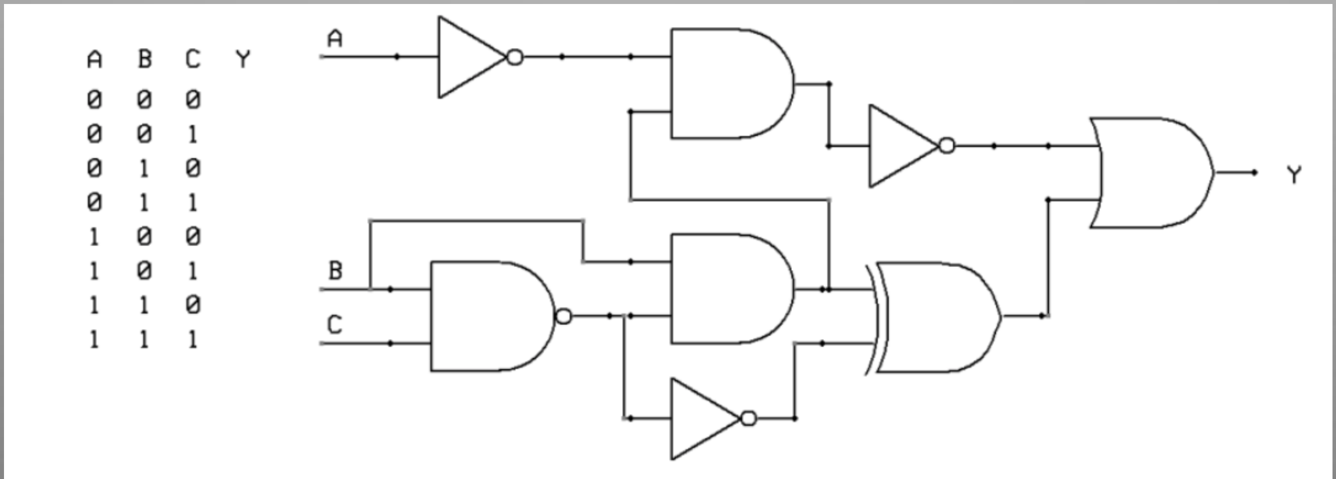
- Each gate in a digital circuit
- Each wire in a digital circuit
- Each “gate” in a noisy circuit, e.g., in a genetic circuit
- Each reaction in a stochastic chemical reaction network

Total entropy flow out of circuit – the thermodynamic cost – is

$$\sum_{g} \left(K\left[p^{pa(g)}(X), q^{pa(g)}(X)\right] - K\left[p^{g}(X), q^{g}(X)\right] + \sum_{c \in L(g)} p^{pa(g)}(c) EP(q^{pa(g);c}) \right)$$

where:

- g indexes the circuit's gates
- pa(g) is parent gates of g in circuit (so $p^{pa(g)}$ is joint distribution into g)
- L(g) is the set of islands of (function implemented by) gate g



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To focus on information theory, assume every $EP(q^{pa(g);c}) = 0$

Total entropy flow out of circuit – the thermodynamic cost – is

$$\sum_g \left(K\left[p^{pa(g)}(X), q^{pa(g)}(X)\right] - K\left[p^g(X), q^g(X)\right] \right)$$

where:

- g indexes the circuit's gates
- pa(g) is parent gates of g in circuit (so $p_0^{pa(g)}$ is joint distribution into g)

- For any circuit C, the "*All-at-once gate*", AO(C), is a single gate that computes same function as C.

Rest of talk: Compare
Landauer costs and total EF for C and AO(C)

Notation:

$$I\left(P(X_1, X_2, \ldots)\right) = \left[\sum_i S\left(P\left(X^i\right)\right)\right] - S\left(P(X_1, X_2, \ldots)\right)$$

- "Multi-information" (a generalization of mutual information)

$$I^D(P, R) = D(P, R) - \left[\sum_i D\left(P\left(X^i\right), R\left(X^i\right)\right)\right]$$

- "KL Multi-information" (a generalization of multi-information based on a "*reference prior*" R)

$$I^K(P, R) = \left[\sum_i K\left(P\left(X^i\right), R(X^i)\right)\right] - K(P, R)$$

- "Cross Multi-information" (multi-information of P minus KL multi-information between P and R)

EF for running circuit C on input distribution p when optimal distribution is q:

$$EF_C(p,q) = \sum_g \left(K_g\left[p^{pa(g)}, q^{pa(g)}\right] - K_g[p^g, q^g]\right)$$

EF for running AO gate that implements same input-output function as C:

$$EF_{AO(C)}(p,q) = K_{in}\left[p^{in}, q^{in}\right] - K_{out}[p^{out}, q^{out}]$$

EF for running circuit C on input distribution p when optimal distribution is q:

$$EF_C(p,q) = \sum_g (K_g\left[p^{pa(g)}, q^{pa(g)}\right] - K_g[p^g, q^g])$$

EF for running AO gate that implements same input-output function as C:

$$EF_{AO(C)}(p,q) = K_{in}\left[p^{in}, q^{in}\right] \text{ - } K_{out}[p^{out}, q^{out}]$$

Thermodynamic penalty / gain by using C rather than AO(C):

$$\begin{aligned}\Delta EF_C(p,q) &= EF_C(p,q) \; - EF_{AO(C)}(p,q) \\ &= I^K(p,q) \; - \sum_g I^K(p^{pa(g)}, q^{pa(g)})\end{aligned}$$

Thermodynamic penalty / gain by using C rather than AO(C):

$$\Delta EF_C(p,q) \;=\; EF_C(p,q) \;-\; EF_{AO(C)}(p,q)$$
$$= I^K(p,q) \;-\; \sum_g I^K(p^{pa(g)}, q^{pa(g)})$$

where $I^K(p, q)$ is cross multi-information

When p = q, ***$\Delta EF_C(p, q)$ cannot be negative***

- Indeed, it can be $+\infty$.

So never an advantage to using a circuit ... if p = q, i.e., if one "guessed right" when designing every single gate.

Thermodynamic penalty / gain by using C rather than AO(C):

$$\Delta EF_C(p,q) = EF_C(p,q) - EF_{AO(C)}(p,q)$$
$$= I^K(p,q) - \sum_g I^K(p^{pa(g)}, q^{pa(g)})$$

So never an advantage to using a circuit, if p = q.

However in real world, *too expensive to build an AO gate*.

Even if p = q, there are circuits where total (Landauer) cost is infinite!

So, even if p = q, if we can't use an AO gate,

what circuit to use to implement a given function**?**

Thermodynamic penalty / gain by using C rather than AO(C):

$$\Delta EF_C(p,q) = EF_C(p,q) \; - EF_{AO(C)}(p,q)$$
$$= I^K(p,q) \; - \textstyle\sum_g I^K(p^{pa(g)}, q^{pa(g)})$$

(Partial) answer: if p = q, extra EF if we use C′ rather than C:

$$\sum_{g \in C'} I(p^{pa(g)}) - \sum_{g \in C} I(p^{pa(g)})$$

I.e., choose circuit with ***smallest multi-informations*** of input distributions into its gates.

However if $p \neq q$, extra *EP* if use C rather than AO(C) is

$$-I^{D}(p,q) + \sum_{g} I^{D}(p^{pa(g)}, q^{pa(g)})$$

This can be positive ***<u>or negative</u>***

When – as in real world – we aren't lucky enough to have gates obey $p = q$, using a circuit C rather than AO(C) may either increase or decrease EP

Even worse: if $p \neq q$, extra *total entropy flow* if use C rather than AO(C) is

$$I^K(p,q) - \sum_g I^K(p^{pa(g)}, q^{pa(g)})$$

This can be positive ***or negative***

In fact, extra EF can be either $-\infty$ or $+\infty$

When – as in real world – we aren't lucky enough to have gates obey $p = q$, using a circuit C rather than AO(C) may either increase or decrease *total entropy flow out of circuit.*

Even worse: if p ≠ q, extra *total entropy flow* if use C rather than AO(C) is

$$I^K(p,q) - \sum_g I^K(p^{pa(g)}, q^{pa(g)})$$

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When – as in real world – we aren't lucky enough to have gates obey p = q, using a circuit C rather than AO(C) may either increase or decrease *total entropy flow out of circuit.*

So, if p ≠ q,

***what circuit to use to implement a given function*?**

OTHER RESULTS

- Some sufficient conditions for *C to have greater EP than AO(C)*
- Some sufficient conditions for *C to have less EP than AO(C)*
- Analysis when *outdegrees of some gates > 1*
- Analysis when *prior distributions q_g at each gate g are arbitrary*.
- Analysis accounting for thermodynamic costs of wires

OTHER RESULTS

- A *family of circuits* refers to any set of circuits that all "implement the same function", just for differing numbers of input bits

  - Circuit complexity theory analyzes how costs (e.g., number of gates) of each circuit in a family of circuits scale with number of input bits.

  - *Extension of circuit complexity theory to include thermodynamics costs*.

- Analysis for "logically reversible circuits" - circuits built out of Fredkin gates, with enough extra gates added to remove all "garbage bits".

CONCLUSIONS

- Exact equations for entire entropy flow of a system:

$$K(p_0, q_0) - K(p_1, q_1) = \text{Landauer cost} + \text{EP}$$

- Different circuits, all implementing the same function, all using thermodynamically reversible gates, have different thermodynamic costs.
- Landauer cost of a circuit C ≥ Landauer cost of AO(C)
- Mismatch cost of a circuit C can be either greater or less than mismatch cost of AO(C). Same for total work of running C vs. AO(C).
- Lots of future research!

D. H. Wolpert and A. Kolchinsky, arXiv:1806.04103 (2018)

Wiki on thermodynamics of computation:

https://centre.santafe.edu/thermocomp

Please visit and start to add material!