The Inverted Pendulum

1 Introduction

The purpose of this lab is to explore the dynamics of the simple harmonic oscillator (SHO). To make things a bit more interesting, we will model and study the motion of an inverted pendulum (IP), which is a special type of tunable mechanical oscillator. As we will see below, the inverted pendulum contains two restoring forces, one positive and one negative. By adjusting the relative strengths of these two forces, we can change the oscillation frequency of the pendulum over a wide range.

As usual (see Chapter 1), we will first make a mathematical model of the inverted pendulum. Then you will characterize the system by measuring various parameters in the model. And finally you will observe the motion of the pendulum and see that it agrees with the model (to within experimental uncertainties).

The inverted pendulum is a fairly simple mechanical device, so you should be able to analyze and characterize the system almost completely. At the same time, the inverted pendulum exhibits some interesting dynamics, and it demonstrates several important principles in physics. Waves and oscillators are everywhere in physics and engineering, and one of the best ways to understand oscillatory phenomenon is to carefully analyze a relatively simple system like the inverted pendulum.

2 Modeling the Inverted Pendulum (IP)

2.1 The Simple Harmonic Oscillator

We begin our discussion with the most basic harmonic oscillator – a mass on a spring. We can write the restoring force $F = -kx$ in this case, where k is the spring constant. Combining this with Newton's law, $F = ma = m\ddot{x}$, gives $\ddot{x} = -(k/m)x$, or

$$\begin{aligned} \ddot{x} + \omega_0^2 x &= 0 \\ \text{with } \omega_0^2 &= k/m \end{aligned} \tag{1}$$

The general solution to this equation is $x(t) = A_1 \cos(\omega_0 t) + A_2 \sin(\omega_0 t)$, where A_1 and A_2 are constants. (You can plug $x(t)$ in yourself to see that it solves the equation.) Once we specify the initial conditions $x(0)$ and $\dot{x}(0)$, we can then calculate the constants A_1 and A_2. Alternatively, we can write the general solution as $x(t) = A\cos(\omega_0 t + \varphi)$, where A and φ are constants.

The math is simpler if we use a complex function $\tilde{x}(t)$ in the equation, in which case the solution becomes $\tilde{x}(t) = \tilde{A}e^{i\omega_0 t}$, where now $\tilde{A}$ is a complex constant. (Again, see that this solves the equation.) To get the actual motion of the oscillator, we then take the real part, so $x(t) = Re[\tilde{x}(t)]$. (If you have not yet covered why this works in your other courses, see the EndNote at the end of this chapter.)

You should be aware that physicists and engineers have become quite cavalier with this complex notation. We often write that the harmonic oscillator has the solution $x(t) = Ae^{i\omega_0 t}$ without specifying what is complex and what is real. This is lazy shorthand, and it makes sense once you become more familiar with the dynamics of simple harmonic motion.

The Bottom Line: Equation 1 gives the equation of motion for a simple harmonic oscillator. The easiest way to solve this equation is using the the complex notation, giving the solution $x(t) = Ae^{i\omega_0 t}$.

2.2 The Simple Pendulum

The next step in our analysis is to look at a simple pendulum. Assume a mass m at the end of a massless string of a string of length ℓ. Gravity exerts a force mg downward on the mass. We can write this force as the vector sum of two forces: a force $mg\cos\theta$ parallel to the string and a force $mg\sin\theta$ perpendicular to the string, where θ is the pendulum angle. (You should draw a picture and see for yourself that this is correct.) The force along the string is exactly countered by the tension in the string, while the perpendicular force gives us the equation of motion

$$\begin{aligned} F_{perp} &= -mg\sin\theta \qquad (2) \\ &= m\ell\ddot{\theta} \\ \text{so } \ddot{\theta} + (g/\ell)\sin\theta &= 0 \end{aligned}$$

As it stands, this equation has no simple analytic solution. However we can use $\sin\theta \approx \theta$ for small θ, which gives the harmonic oscillator equation

$$\ddot{\theta} + \omega_0^2\theta = 0 \qquad (3)$$

$$\text{where } \omega_0^2 = g/\ell \qquad (4)$$

The Bottom Line: A pendulum exhibits simple harmonic motion described by Equation 3, but only in the limit of small angles.

2.3 The Simple Inverted Pendulum

Our model for the inverted pendulum is shown in Figure xxx. Assuming for the moment that the pendulum leg has zero mass, then gravity exerts a force

$$\begin{aligned} F_{perp} &= +Mg\sin\theta \qquad (5) \\ &\approx Mg\theta \end{aligned}$$

where F_{perp} is the component of the gravitational force perpendicular to the leg, and M is the mass at the end of the leg. The force is positive, so gravity tends to make the inverted pendulum tip over, as you would expect.

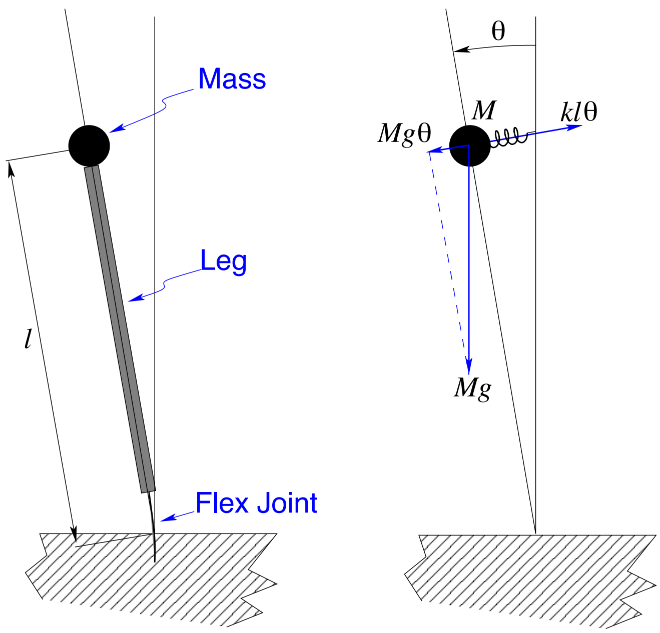

Figure 1: Simple inverted pendulum

In addition to gravity, we also have a flex joint at the bottom of the leg that is essentially a spring that tries to keep the pendulum upright. The force from this spring is given by Hooke's law, which we can write

$$\begin{aligned} F_{spring} &= -kx \\ &= -k\ell\theta \end{aligned} \tag{6}$$

where ℓ is the length of the leg.

The equation of motion for the mass M is then

$$\begin{aligned} M\ddot{x} &= F_{perp} + F_{spring} \\ M\ell\ddot{\theta} &= Mg\theta - k\ell\theta \end{aligned} \tag{7}$$

and rearranging gives

$$\ddot{\theta} + \omega_0^2\theta = 0 \tag{8}$$

$$\text{where } \omega_0^2 = \frac{k}{M} - \frac{g}{\ell} \tag{9}$$

If ω_0^2 is positive, then the inverted pendulum exhibits simple harmonic motion $\theta(t) = Ae^{i\omega_0 t}$. If ω_0^2 is negative (for example, if the spring is too weak, or the top mass is to great), then the pendulum simply falls over.

The Bottom Line: A simple inverted pendulum (IP) exhibits simple harmonic motion described by Equation 8. The restoring force is supplied by a spring at the bottom of the IP, and there is also a negative restoring force from gravity. The resonant frequency can be tuned by changing the mass M on top of the pendulum.

2.4 A Better Model of the Inverted Pendulum

The simple model above is unfortunately not good enough to describe the real inverted pendulum in the lab. We need to include a nonzero mass m for the leg. In this case it is best to start with Newton's law in angular coordinates

$$I_{tot}\ddot{\theta} = \tau_{tot} \tag{10}$$

where I is the total moment of inertia of the pendulum about the pivot point and τ is the sum of all the relevant torques. The moment of inertia of the large mass is $I_M = M\ell^2$, while the moment of inertia of a thin rod pivoting about one end (you can look it up, or calculate it) is $I_{leg} = m\ell^2/3$. Thus

$$\begin{aligned} I_{tot} &= M\ell^2 + \frac{m\ell^2}{3} \\ &= \left(M + \frac{m}{3}\right)\ell^2 \end{aligned} \tag{11}$$

The torque consists of three components

$$\begin{aligned} \tau_{tot} &= \tau_M + \tau_{leg} + \tau_{spring} \\ &= Mg\ell\sin\theta + mg\left(\frac{\ell}{2}\right)\sin\theta - k\ell^2\theta \end{aligned} \tag{12}$$

The first term comes from the usual expression for torque $\tau = r \times F$, where F is the gravitational force on the mass M, and r is the distance between the mass and the pivot point. The second term is similar, using $r = \ell/2$ for the center-of-mass of the leg. The last term derives from $F_{spring} = -k\ell\theta$ above, converted to give a torque about the pivot point.

Using $\sin\theta \approx \theta$, this becomes

$$\begin{aligned} \tau_{tot} &\approx Mg\ell\theta + mg\left(\frac{\ell}{2}\right)\theta - k\ell^2\theta \\ &\approx \left[Mg\ell + \frac{mg\ell}{2} - k\ell^2\right]\theta \end{aligned} \tag{13}$$

and the equation of motion becomes

$$I_{tot}\ddot{\theta} = \tau_{tot} \tag{14}$$

$$\left(M + \frac{m}{3}\right)\ell^2\ddot{\theta} = \left[Mg\ell + \frac{mg\ell}{2} - k\ell^2\right]\theta \tag{15}$$

which is a simple harmonic oscillator with

$$\omega_0^2 = \frac{k\ell^2 - Mg\ell - \frac{mg\ell}{2}}{\left(M + \frac{m}{3}\right)\ell^2} \tag{16}$$

This expression gives us an oscillation frequency that better describes our real inverted pendulum. If we let $m = 0$, you can see that this becomes

$$\omega_0^2(m = 0) = \frac{k}{M} - \frac{g}{\ell} \tag{17}$$

which is the frequency of the simple inverted pendulum described in the previous section. If we remove the top mass entirely, so that $M = 0$, you can verify that

$$\omega_0^2(M = 0) = \frac{3k}{m} - \frac{3g}{2\ell} \tag{18}$$

The Bottom Line: The math gets a bit more complicated when the leg mass m is not negligible. The resonance frequency of the IP is then given by Equation 16. This reduces to Equation 17 when $m = 0$, and to Equation 18 when $M = 0$.

3 The Damped Harmonic Oscillator

To describe our real pendulum in the lab, we will have to include damping in the equation of motion. One way to do this (there are others) is to use a complex spring constant given by

$$\tilde{k} = k(1 + i\phi) \tag{19}$$

where k is the normal (real) spring constant and ϕ (also real) is called the *loss angle*. Looking at a simple harmonic oscillator, the equation of motion becomes

$$m\ddot{x} = -k(1 + i\phi)x \tag{20}$$

which we can write

$$\ddot{x} + \omega_{damped}^2 x = 0 \tag{21}$$

$$\text{with } \omega_{damped}^2 = \frac{k(1 + i\phi)}{m} \tag{22}$$

If the loss angle is small, $\phi \ll 1$, we can do a Taylor expansion to get the approximation

$$\omega_{damped} = \sqrt{\frac{k}{m}}(1 + i\phi)^{1/2} \tag{23}$$

$$\approx \sqrt{\frac{k}{m}}(1 + i\frac{\phi}{2}) \tag{24}$$

$$= \omega_0 + i\alpha \tag{25}$$

$$\text{with } \alpha = \frac{\phi\omega_0}{2} \tag{26}$$

Putting all this together, the motion of a weakly damped harmonic oscillator becomes

$$\tilde{x}(t) = \tilde{A}e^{-\alpha t}e^{i\omega_o t} \tag{27}$$

which here $\tilde{A}$ is a complex constant. If we take the real part, this becomes

$$x(t) = Ae^{-\alpha t}\cos(\omega_0 t + \varphi) \tag{28}$$

This is the normal harmonic oscillator solution, but now we have the extra $e^{-\alpha t}$ term that describes the exponential decay of the motion.

We often refer to the *quality factor* Q of an oscillator, which is defined as

$$Q = \omega \frac{\text{Energy stored}}{\text{Power loss}} \tag{29}$$

Note that Q is a dimensionless number. For our case this becomes (the derivation is left for the reader)

$$Q \approx \frac{\omega_0}{2\alpha} = \frac{1}{\phi} \tag{30}$$

The Bottom Line: We can model damping in a harmonic oscillator by introducing a complex spring constant. Solving the equation of motion then gives damped oscillations, given by Equations 27 and 28 when the damping is weak.

4 The Driven Harmonic Oscillator

If we drive a simple harmonic oscillator with an external oscillatory force, then the equation of motion becomes

$$\ddot{x} + \omega_{damped}^2 x = \frac{F_0}{m}e^{i\omega t} \tag{31}$$

where ω is the angular frequency of the drive force and F_0 is the applied force. (As above, $\omega_{damped} = \omega_0 + i\alpha$.) Analyzing this shows that the system first exhibits a transient behavior that lasts a time of order

$$t_{transient} \approx \alpha^{-1} \approx 2Q/\omega_0 \tag{32}$$

During this time the motion is quite complicated, depending on the initial conditions and the phase of the applied force.

The transient behavior eventually dies away, however, and for $t \gg t_{transient}$ the system settles into a *steady-state* behavior, where the motion is given by

$$x(t) = Xe^{i\omega t} \tag{33}$$

In other words, in steady-state the system oscillates with the same frequency as the applied force, regardless of the natural frequency ω_0. Plugging this $x(t)$ into the equation of motion quickly gives us

$$X = \frac{F_0/m}{\omega_{damped}^2 - \omega^2} \tag{34}$$

Since X is a complex constant, it gives the amplitude and phase of the motion. When the damping is small ($\alpha \ll \omega_0$), we can write $\omega_{damped}^2 \approx \omega_0^2 + 2i\alpha\omega_0$, giving

$$X \approx \frac{F_0/m}{(\omega_0^2 - \omega^2) + 2i\alpha\omega_0} \tag{35}$$

and the amplitude of the driven oscillations becomes

$$|X(\omega)| = \frac{|F_0/m|}{\sqrt{\left(\omega_0^2 - \omega^2\right)^2 + 4\alpha^2\omega_0^2}} \tag{36}$$

Note that the driven oscillations have the highest amplitude on resonance ($\omega = \omega_0$), and the peak amplitude is highest when the damping is lowest.

The Bottom Line: Once the transient motions have died away, a harmonically driven oscillator settles into a steady-state motion exhibiting oscillation at the same frequency as the drive. The amplitude is highest on resonance (when $\omega = \omega_0$) and when the damping is weak, as given by Equation 36.

5 The Transfer Function

For part of this lab you will shake the base of the inverted pendulum and observe the response. To examine this theoretically, we can look first at the simpler case of a normal pendulum in the small-angle approximation (when doing theory, always start with the simplest case and work up). The force on the pendulum bob (see Equation 2) can be written

$$F = -mgx/\ell \tag{37}$$

where x is the horizontal position of the pendulum and ℓ is the length. If we shake the top support of the pendulum with a sinusoidal motion, $x_{top} = X_{drive}e^{i\omega t}$, then this becomes

$$\begin{aligned} F &= -mg\left(x - x_{top}\right)/\ell && (38) \\ \ddot{x} + \omega_0^2 x &= \omega_0^2 x_{top} && (39) \end{aligned}$$

where $\omega_0^2 = g/\ell$. With damping this becomes

$$\begin{aligned} \ddot{x} + \omega_{damped}^2 x &= \omega_0^2 x_{top} && (40) \\ &= \omega_0^2 X_{drive} e^{i\omega t} && (41) \end{aligned}$$

which is essentially the same as Equation 31 for a driven harmonic oscillator. From the discussion above, we know that this equation has a steady-state solution with $x = Xe^{i\omega t}$. It is customary to define the *transfer function*

$$H(\omega) = \frac{X}{X_{drive}} \tag{42}$$

which in this case is the ratio of the motion of the pendulum bob to the motion of the top support. Since H is complex, it gives the ratio of the amplitudes of the motions and their relative phase.

For the simple pendulum case, Equation 35 gives us (verify this for yourself)

$$H(\omega) \approx \frac{\omega_0^2}{(\omega_0^2 - \omega^2) + 2i\alpha\omega_0} \tag{43}$$

At low frequencies ($\omega \ll \omega_0$) and small damping ($\alpha \ll \omega_0$), this becomes $H \approx 1$, as you would expect (to see this, consider a mass on a string, and shake the string as you hold it in your hand). At high frequencies ($\omega \gg \omega_0$), this becomes $H(\omega) \approx -\omega_0^2/\omega^2$, so the motion of the bob is 180 degrees out of phase with the motion of the top support (try it).

The Bottom Line: The transfer function gives the complex ratio of two motions, and it is often used to characterize the behavior of a driven oscillator. Equation 43 shows one example for a simple pendulum. The motion of the inverted pendulum is a bit more interesting, as you will see when you measure $H(\omega)$ in the lab.

6 The Inverted Pendulum Test Bench

6.1 Care and Use of the Apparatus

The Inverted Pendulum hardware is not indestructible, so please treat it with respect. Ask your TA if you think something is broken or otherwise amiss. The flex joint is particularly delicate, and bending it to large angles can cause irreparable damage. Follow these precautions:

1. Never let the IP oscillate without the travel limiter.
2. Do not let the IP leg fall.
3. Do not disassemble the IP without assistance from your TA.

7 The Lab - First Week

7.1 Pre-Lab Problems

1. Rewrite Equation 16 using the "angular stiffness" variable $\kappa = k\ell^2$ for the spring constant. What are the units of k, κ?

2. Determine the length of an IP leg with a load of $M = 0.383$ kg, flex joint angular stiffness $\kappa = 2.5$ Nm, oscillation period $T = 10$ seconds, and negligible leg mass, $m = 0$. Compute the length of a simple pendulum with the same oscillation frequency.

3. For the IP in Problem 2, calculate the mass difference ΔM needed to change the period from 10 seconds to 100 seconds. At what mass does the period go to infinity?

4. For the IP in Problem 2, if the loss angle is $\phi = 10^{-2}$, how long does it take for the motion to damp to 1% of its starting amplitude?

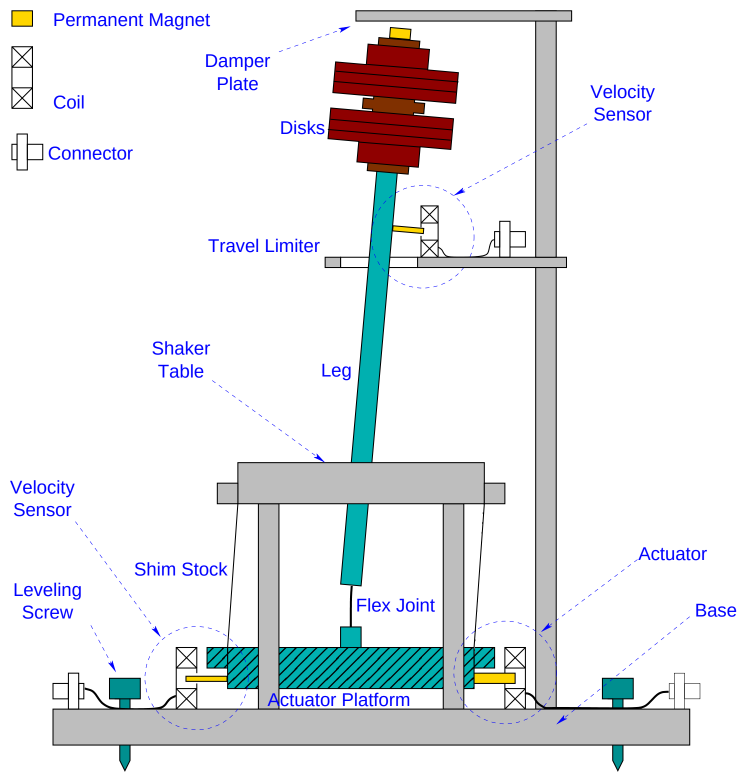

Figure 2: Inverted-pendulum test bench.

7.2 In-Lab Exercises

7.2.1 Getting Started

Please read down to the end of this section before beginning your work. Once you begin in the lab, record all your data and other notes in your notebook as you proceed. Print out relevant graphs and tape them into your notebook as well.

Step 1. All the measurements this week are done with the actuator platform locked in place. Use the attached thumb screws to lock the platform (see your TA if you are not sure about this). If the platform is securely locked, it should not rattle if you shake it gently.

7.2.2 The IP Leg

When adding mass to the top of the leg, add weights symmetrically on the load device (see Figure 3). This ensures that the center-of-mass of the added weight

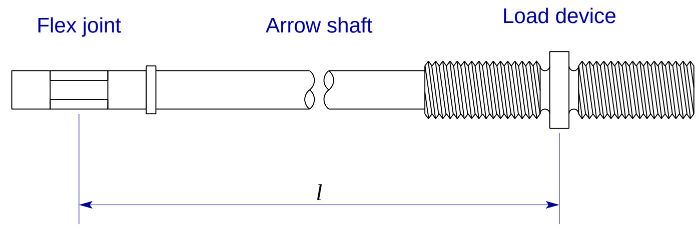

Figure 3: IP leg with flex joint and load device

is always positioned at the same distance from the pivot point (i.e., this ensures that ℓ stays constant as you change M).

Step 2. Using the spare leg in the lab, measure the length ℓ and mass m of the leg, including measurement uncertainties (error bars). Measure ℓ from the center of the flex joint to the center-of-mass of the added weight. Note that you cannot actually measure ℓ very well because of the large size of the flex joint. Use common sense to estimate an uncertainty in ℓ, based on the length of the flex joint and how accurately the load mass can be placed. Note that your measurement of m does not include the mass of the magnet assembly on the IP, which you can take to be $M_{magent} = 6$ grams.

7.2.3 Resonant Frequency versus Load

Step 3. With no mass on the top of the leg ($M_{add} = 0$), measure the oscillation frequency $P = \omega_1/(2\pi)$ (with an error bar) of the IP using an oscilloscope and the sensor outputs provided. Use Equation 16 to estimate k based on your frequency measurement. Assume $M = M_{add} + M_{magnet} = M_{magnet}$ for this calculation, and use your direct measurements of m and ℓ. Assuming $M_{magnet} = 6 \pm 1$ grams, and that the theory in Equation 16 is exact, use standard error propagation methods to estimate an error bar for k from the errors in the other quantities.

Step 4. Make additional measurements of P for different values of M_{add}. Use the load device with no washers, then add washers symmetrically about the center-of-mass (see Figure 3). For each measurement of P, also measure M_{add} using the scale in the lab. Take at least 5-6 data points with error bars. Note that the errors in the measurement of P are larger for larger P. Make a plot of $P(M_{add})$, and add the plot to your notebook. If you don't have your own computer with a data-plotting program on it, there is an iMac in the lab with *Mathematica*, *Kaleidagraph*, and Microsoft *Excel* on it that you can use. You can use graph paper and a pen if you like, but if you are planning on doing much science in the future, we strongly encourage you to learn and get practice with computer data-analysis tools such as *Mathematica* and *Kaleidagraph*.

Note that you may have to level the IP as you add more mass (using the three leveling screws; ask your TA if you need help). As the top mass increases, the

IP is more likely to tip over, so the leveling of the apparatus is more important.

Step 5. Now use Equation 16 to create a theory line to add to your data plot.

Try putting in different values of k to see how the theory changes. You should get a nice fit using your calculated value of k, although a different value may fit better. The real IP is not exactly the same as depicted in theory, so there will be some systematic discrepancies. Your data may also deviate significantly from theory as the calculated period goes to infinity (because the real IP is not perfect). Plot the data with theory for your best guess for k, and add the plot to your notebook.

Step 6. From observing how well the theory fits the data for different guesses for k, estimate your best-fit k along with an error bar. How does this compare with your estimate of k in Step 3?

8 The Lab - Second Week

8.1 Pre-Lab Problems

1. Plot the amplitude response of a driven harmonic oscillator, given by Equation 36. Assume a resonance frequency $\nu_0 = 1$ Hz, $F_0/m = 1$, and plot the amplitude as a function of frequency $\nu = \omega/2\pi$. Make three plots (preferably all on a single graph) using $Q = 1, 10, 100$. Plot all three on a linear-linear plot, then plot all three again on a log-log plot. Note that the response shows a power-law behavior ($x \sim \nu^{-2}$) at high frequencies, and this appears as a straight line in the log-log plots.

2. Let $\Delta\nu$ be the FWHM (full-width at half-maximum) width of $|A(\nu)|$ (i.e. when $\nu = \nu_0 \pm \Delta\nu/2$, then $|A(\nu)| = |A(\nu_0)|/2$). In the limit of high Q, what is $\Delta\nu$ as a function of Q? (Use Equation 36 and let $\omega = \omega_0 + \varepsilon$, so near the peak you can take $\omega^2 \approx \omega_0^2 + 2\omega_0\varepsilon$.)

3. In the lab we will measure $\dot{X}(\nu)$ and $\dot{X}_{drive}(\nu)$, the velocities of the pendulum bob and support platform. Show that $H(\nu) = X(\nu)/X_{drive}(\nu) = \dot{X}(\nu)/\dot{X}_{drive}(\nu)$.

8.2 In-Lab Exercises

8.2.1 Inverted Pendulum Loss Angle

Step 1. With the actuator platform locked (same as last week), start the pendulum oscillating, and observe its motion with no added mass and no added damping. With a long enough timebase on your oscilloscope, you should be able to observe the amplitude of the oscillations decrease, and from that you can estimate the ringdown time. The oscilloscope has a "print" button by the lower left corner of the screen. Push that, and you should get a printout of your screen at the printer. Put a plot of the IP motion $x_{IP}(t)$ (your screenshot) in your notebook, and estimate the loss angle ϕ from the time it takes the motion to damp away. Now add the aluminum damper plate to the top of the IP assembly (ask your TA). As the pendulum swings, the damping magnet induces a current in the aluminum plate, which heats the plate and extracts energy from the IP motion. With the aluminum plate in place (about 6 mm from the magnet), again plot $x_{IP}(t)$ and estimate ϕ. Unscrew the magnet holder tube a bit so the magnet is quite close to the plate, and again plot $x_{IP}(t)$ and estimate ϕ. For each case, how long is $t_{transient}$ – the time needed for a driven system to reach steady-state motion (see Equation 32)?

8.2.2 The Transfer Function

The rest of the lab is done with the actuator platform unlocked; ask your TA if you need help with this. Drive the motion of the platform using a sinusoidal voltage from the signal generator. Be sure to set the "sweep" button to EXT (which turns *off* the sweep feature of the signal generator). With a high signal amplitude you should be able to see the platform oscillating, and you can see the frequency change as you change the frequency of the drive voltage.

With no added mass on the IP, use the oscilloscope to measure the motion of the platform $x_{platform}(t)$ (channel 1) and the motion of the top mass $x_{IP}(t)$ (channel 2). If both do not show simple sinusoidal oscillations, turn down the drive amplitude. Measure the amplitude and relative phase of both as you change the drive frequency. Divide $x_{IP}(t)$ by $x_{platform}(t)$ to get the amplitude of the transfer function as a function of frequency.

Step 2. Collect data at a sufficient number of drive frequencies to map out the transfer function $H(\nu)$ as a function of frequency. Put a plot of $H(\nu)$ in your notebook, and qualitatively explain the features of $H(\nu)$ (both amplitude and phase), given the above discussion of a driven harmonic oscillator.

Step 3. Now add mass to the IP so the resonant frequency is between 0.5 and 1 Hz (use your data from last week to see what M is needed). Place the damper plate about 3 mm from the damping magnet. Starting at low frequencies, and again map out $H(\nu)$. Again, make sure that both motions are sinusoidal. Above the resonance frequency of the IP, you will need to turn up the drive amplitude so the signal-to-noise in the measurements remains adequate. Be sure to continue your measurements to at least 10 Hz. Put a plot of $H(\nu)$ in your notebook, and again qualitatively explain the features of $H(\nu)$ (both amplitude and phase). [Hint: an understanding of the "center of percussion" of a bar will be useful for explaining the high-frequency behavior.]

9 EndNote: Using Complex Functions to Solve Real Equations

Physicists and engineers often use complex functions to solve real equations, with the understanding that you take the real part at the end. Why does this work? And why do we even do this? We can demonstrate with the simple harmonic oscillator. Start with the equation of motion $\ddot{x} + \omega_0^2 x = 0$, and let us solve this using a complex function: $x = \alpha + i\beta$, where $\alpha(t)$ and $\beta(t)$ are real functions. If you plug this in, you will see that $\ddot{x} + \omega_0^2 x = 0$ becomes

$$\begin{aligned} \left(\ddot{\alpha} + i\ddot{\beta}\right) + \omega_0^2 \left(\alpha + i\beta\right) &= 0 \\ \left(\ddot{\alpha} + \omega_0^2 \alpha\right) + i\left(\ddot{\beta} + \omega_0^2 \beta\right) &= 0 \end{aligned} \tag{44}$$

Since a complex number equals zero only if both the real and imaginary parts equal zero, we see that $\ddot{x} + \omega_0^2 x = 0$ implies that both $\ddot{\alpha} + \omega_0^2 \alpha = 0$ and $\ddot{\beta} + \omega_0^2 \beta = 0$. In other words, both the real and imaginary parts of $x(t)$ satisfy the original equation.

So we have a procedure: try using a complex function to solve the original equation. If this works, then taking the real part of the solution gives a real function that also solves the same differential equation. (If in doubt, then verify directly that the real part solves the equation.)

Why do we go to the trouble of using complex functions to solve a real equation? Because differential equations are often easier to solve when we assume complex functions (seems counterintuitive, but it's true). The function $e^{i\omega_0 t}$ is a simple exponential, and the derivative of an exponential is another exponential – that makes things simple. In contrast, cosines and sines are more difficult to work with.

In the case of the simple harmonic oscillator, the solution $x(t) = Ae^{i\omega t}$ has a natural interpretation. The length and angle of the A vector (in the complex plane) give the amplitude and phase of the oscillations.

You should note, however, that this only works for linear equations. If our equation were $\ddot{x} + \omega_0^2 x + \gamma x^2 = 0$, for example, then using complex functions would not have the same benefits. In fact there is no simple solution to this equation, complex or otherwise. This equation describes a nonlinear oscillator, and nonlinear oscillators exhibit a fascinating dynamics with interesting behaviors that people still study to this day.