CALIFORNIA INSTITUTE OF TECHNOLOGY
PHYSICS MATHEMATICS AND ASTRONOMY DIVISION

Freshman Physics Laboratory (PH003)

Classical Mechanics
The Maxwell Top

Chapter 2

The Maxwell Top

2.1 Introduction

In this chapter we want to study some particular cases of rigid body dynamics, which have a rotational symmetry around one axis, the so-called top, or Maxwell top.[1]

In general, to solve the dynamics of a rigid body we must apply the second law of dynamics, i.e.

$$\frac{d\vec{L}}{dt} = \vec{\tau},$$

where $\vec{\tau}$ is the external torque acting on the body, and $\vec{L}$ is its angular momentum. For a solid body rotating around one of its axis of symmetry $\hat{z}$ (more generally, around any of its three principal axes), with angular velocity $\dot{\theta}$, $\vec{L}$ is given by[2]

$$\vec{L} = I\dot{\theta}\hat{z}, \tag{2.1}$$

where I is the moment of inertia around the $\hat{z}$ axis. I is

$$I = \int (x^2 + y^2) dm,$$

where x and y are the coordinates of the mass dm. In this particular case,

[1]The study of the top general equations of motion is quite complicated and is one of the main topics of a classical mechanics course.

[2]The dot above the symbol stands for the derivative with respect to the time t. The number of dots indicates the order of derivation.

the second law of dynamics assumes the simpler form

$$I\ddot{\theta}\hat{z} = \vec{\tau}.$$

2.2 Some Relevant Examples

In this section we will study three particular cases of the Maxwell top dynamics, which will be used in the laboratory procedures.

2.2.1 Angular Acceleration under a Constant Torque

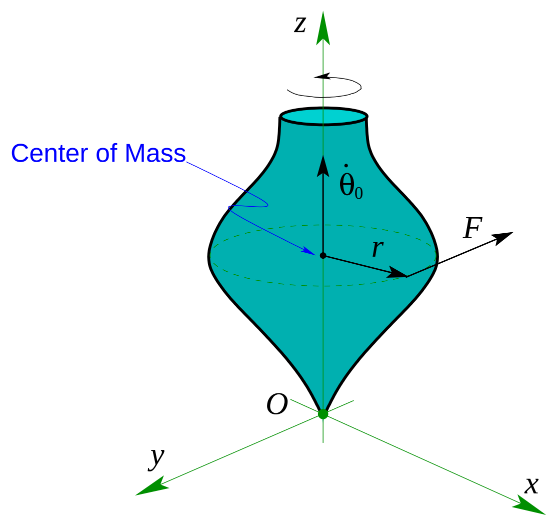


Figure 2.1: Top subject to an external force $\vec{F}$ which produces a torque $\vec{\tau} = rF\hat{z}$.

Let's consider a top, whose axis of symmetry is vertical, and a force $\vec{F}$ applied tangent to the top's surface in the horizontal plane containing the top's center of mass (see figure 2.1). If $\vec{F}$ remains constant in modulus and direction in the reference frame rotating with the top, the second law of dynamics assumes a very simple form, i.e.[3]

$$I\ddot{\theta} = rF,$$

[3]we are neglecting the energy dissipation mechanisms, which are always present in any physical system.

where r is the arm lever distance. Integrating the previous equation we get

$$\theta(t) = \theta_0 + \dot{\theta}_0 t + \frac{1}{2}\ddot{\theta}_0 t^2, \qquad \ddot{\theta}_0 = \frac{rF}{I}. \tag{2.2}$$

where θ_0 is the initial angle, $\dot{\theta}_0$ the initial angular velocity, and $\ddot{\theta}_0$ is the angular acceleration which is also constant.

2.2.2 Top Suspended with a Torsional Rod

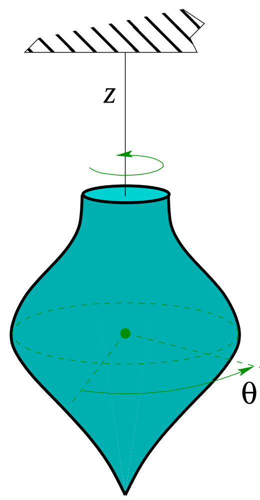


Figure 2.2: Top suspended to a torsional rod.

By suspending the top with a torsional rod (see figure 2.2) we will have a restoring torque (the torsional version of the Hooke's law) given by the linear equation

$$\tau = -k\theta,$$

where θ is the angle in the horizontal plane measured from the equilibrium position. From the second law of dynamics, we will have

$$I\ddot{\theta} = -k\theta,$$

which is the equation of an harmonic oscillator, whose general solution is

$$\theta(t) = \theta_0 \cos(\omega_0 t + \varphi_0), \qquad \omega_0^2 = \frac{k}{I}.$$

The top will oscillate sinusoidally around the vertical axis with angular frequency ω_0. The constant ω_0 is said to be the angular resonant frequency of the torsional pendulum.

2.2.3 Precession of the Top

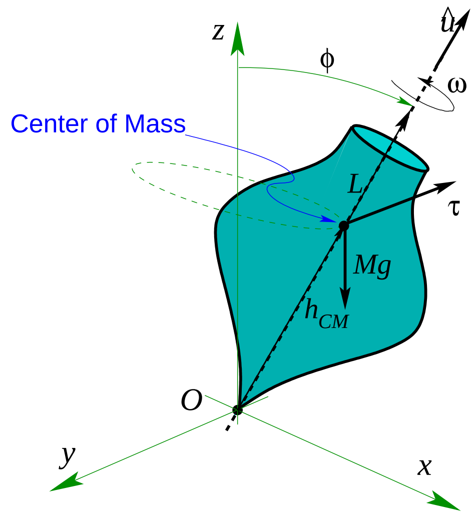


Figure 2.3: Precession of the top. In this sketch $\vec{\tau}$ is parallel to the y axis and is pointing to the negative direction of the y axis. The vector $\vec{h_{CM}}$ is in the plane Oxz

Let's suppose now that the top has its tip constrained on a horizontal plane and is rotating around its axis $\hat{u}$ at a constant angular velocity $\omega\hat{u}$ (see figure 2.3).

If the rotation axis makes an angle ϕ with $\hat{u}$ axis, the modulus of the torque $\vec{\tau}$, due to the gravity force $M\vec{g}$, is

$$\tau = |\vec{h}_{CM} \times M\vec{g}| = h_{CM} M g \sin\phi, \tag{2.3}$$

where $\vec{h}_{CM}$ is the vector pointing to the top center of mass. Because $\vec{g}$ is always vertical, $\vec{\tau}$ must always lie in the horizontal plane.

Because $\omega\hat{u}$ is parallel to $\vec{h}_{CM}$ at all times $\vec{L}$ is parallel to $\vec{h}_{CM}$ also. It follows that $\vec{L}$ is perpendicular to $\vec{\tau}$ at all times.

For the second law of dynamics and because $\vec{\tau}$ is always in the horizontal plane, the variation $d\vec{L}$ of the angular momentum must be always in the horizontal plane. This implies that projection of $\vec{L}$ along the vertical axis is constant and $\vec{L}$ can only rotate about the vertical axis. As a consequence the component of $\vec{L}$ in the horizontal plane is constant in modulus but not in direction.

Considering that the projection of $\vec{L}$ in the horizontal plane is $L\sin\phi = const.$, the variation dL of $\vec{L}$ must be (see figure2.4)

$$dL = L\sin\phi d\alpha,$$

where $d\alpha$ is the infinitesimal angular variation in the horizontal plane.

Using the second law of dynamics and the previous expression, we get

$$L\sin\phi\frac{d\alpha}{dt} = \tau,$$

The derivative is indeed the angular velocity Ω of the top around the vertical axis. Combining the previous expression with the (2.3) we get

$$h_{CM}Mg = L\Omega.$$

Substituting the (2.1) into the previous equation ($\dot{\theta} = \omega$) we finally get

$$\Omega = \frac{Mg}{I\omega}h_{CM}, \tag{2.4}$$

which shows that Ω does not depend on the angle ϕ. Ω is said to be the precession angular frequency and when $\Omega \neq 0$, the top is said to precess around the vertical axis. It is worthwhile to notice that the angular momentum modulus $|\vec{L}|$ is conserved.

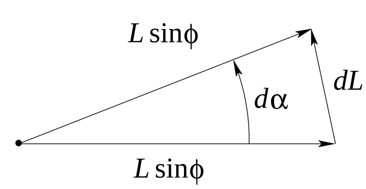


Figure 2.4: Projection and variation of the angular momentum in the horizontal plane

2.3 Experimental setup.

The Maxwell top is shown schematically in Fig.2.5. The top floats on an air cushion which creates a thin "air film" (less than 80μm) and considerably reduces the frictional losses of energy. The two drive jets give the top a small torque, which can be changed acting on the adjustable exhaust valve. The sliding mass m changes the position of the top center of mass along the top axis. Fig.2.5 also shows some details of the air circuit which sustains the top, and creates a thin air film for friction reduction between the base and the top.

Some other instruments needed for the two-week experiment are the following:

- a balance to measure various masses,
- a tachometer to measure the top angular velocity about its axis,
- a ring to increase the top moment of inertia,
- a torsional rod to suspend the top,
- a quasi-frictionless pulley, and a 2g weight to apply a constant torque to the top.

2.3.1 Care and Use of the Experimental Apparatus

The air bearing is a particularly delicate device because of the the air film thickness. Any scratch or dirt on the air bearing surfaces can compromise the use of the experimental apparatus.

These are the precautions that need to be taken:

- TURN THE AIR SUPPLY TO **26PSI** BEFORE ANY OPERATION.
- NEVER LET THE TOP SIT ON THE AIR BEARING BASE WITHOUT AIR FLOW.
- DO NOT SWITCH TOPS. EACH TOP WORKS PROPERLY WITH JUST ONE BASE.
- DO NOT LET ANY OBJECT FALL DOWN INTO THE AIR BEARING CUP.

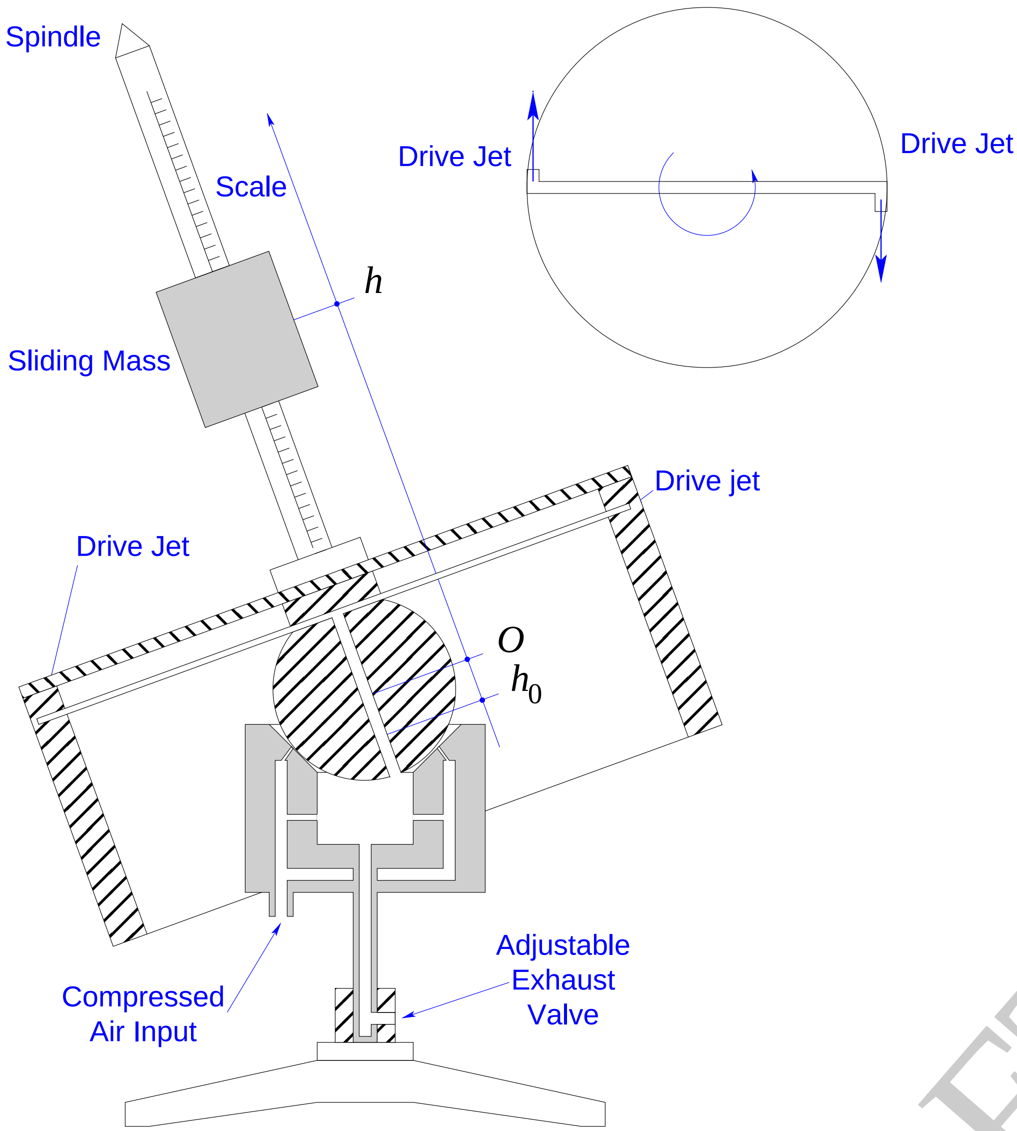


Figure 2.5: Maxwell Top schematic vertical cross section and top view cross section. O is the top's pivoting point and the sketched axis is oriented as indicated by the arrow. This implies that h_0 is negative and h is positive.

- DO NOT USE CLEAR SCOTCH TAPE TO ATTACH WIRES TO TOP'S CYLINDER. USE SCOTCH MASKING TAPE PROVIDED BY THE LABORATORY.
- ALWAYS REMOVE THE SCOTCH TAPE FROM THE TOP'S CYLINDER ONCE FINISHED.
- REMEMBER TO CLOSE THE AIR SUPPLY OUTPUT ONCE FINISHED.

2.4 First Laboratory Week

The purpose of the lab is to apply the two methods of measuring the moment of inertia, based on the theory explained in the previous sections, and compare and analyze the results.

2.4.1 Indirect Measurement of the Moment of Inertia Applying a Constant Torque

The top's moment of inertia I can be indirectly measured if we apply a known constant torque rF to it, and measure the revolution time of the top.

In fact, measuring the elapsed time for $1, 2, 3, ..., n$ revolutions, and fitting the data to the equation (2.2), we can obtain the value of the parameter $\ddot{\theta}_0$ and indeed I.

2.4.2 Indirect Measurement of the Moment of Inertia Using a Torsional Pendulum

Another way to make an indirect measurement of a rigid body moment of inertia I is measuring the period T of a torsional pendulum, whose bob is the rigid body itself. By adding to the bob another rigid body with the known moment of inertia I_0 and re-measuring T, it allows to compute I without knowing the characteristic of the torsional rod.

In fact, the angular frequencies of the rotation about the axis of symmetry for the two cases are

$$\omega_1^2 = \frac{k}{I}, \qquad \omega_2^2 = \frac{k}{I + I_0},$$

which combined together, and considering that $\omega_i = 2\pi/T_i$, result in

$$I = \frac{I_0}{\left(\frac{T_2}{T_1}\right)^2 - 1} \tag{2.5}$$

The value of I_0can be obtained indirectly by the definition of moment inertia.

2.4.3 Propaedeutic Problems

1. Derive how the moment of inertia I_0 for a ring of inner radius r, outer radius R, and mass M, is given by

$$I_0 = \frac{1}{2}M(R^2 + r^2) \tag{2.6}$$

2. If the ring has a mass $M = (5.000 \pm 0.003)$ kg, an outer diameter $D = (200.0 \pm 0.5)$ mm, and an inner diameter $d = (180.0 \pm 0.5)$ mm, compute I_0 ,$\sigma_{I_0}$and the relative error σ_{I_0}/I_0.

3. The measurement of the oscillation period of a torsional pendulum with a stopwatch, produces an error due to the experimenter's reaction time. Assuming that this error is $\sigma = 0.05$ s, the pendulum period is $T = 2$ s, and only one measurement is performed, how many periods must be measured to get a relative error σ_T/T of ± 2%, ± 1%, and ± 0.1%?

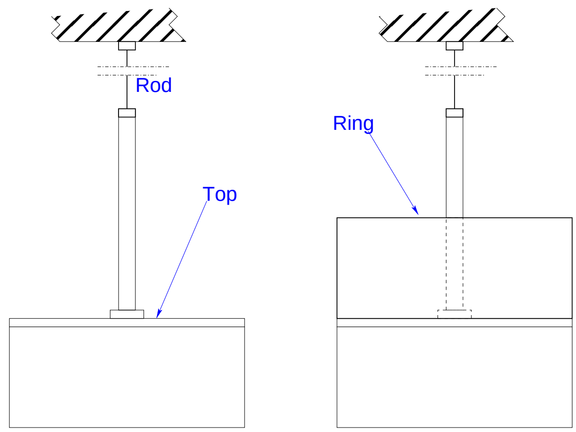
4. In determining the top's moment of inertia I with the torsional pendulum, it is found that the oscillation period is $T_1 = (1.260 \pm 0.003)$ s, and with the added ring with moment of inertia I_0, is $T_2 = (1.750 \pm 0.002)$ s. Find the uncertainty in the measurement of I. Use the values of I_0 and σ_{I_0}given in problem 2.

5. REMEMBER TO CLOSE THE AIR SUPPLY OUTPUT ONCE FINISHED.

2.4.4 Procedure (Top's Moment of Inertia Measurements)

Remember to follow the directives written in section 2.3.1 (Care and Use of the Experimental Apparatus) before starting the procedure.

1. Determine the top moment of inertia I using the torsional rod as show in the figure below.

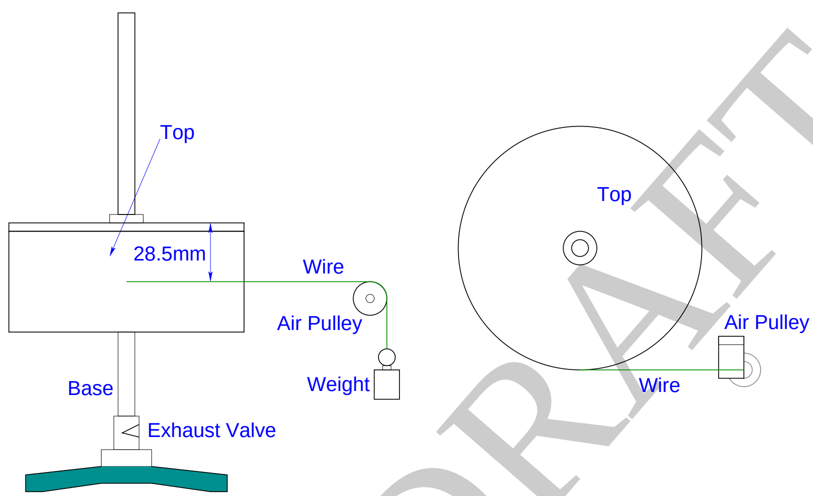


Use the provided reflective sensor and the matlab command MTOscPeriod to measure the torsional period. For example, to measure the period by averaging the periods measured during 20 s, type

MTOscPeriod(20);

Type help MTOscPeriod for a more complete command description and usage.

2. Determine the top's moment of inertia fitting the angular displacement v.s. elapsed time, when the top is under a constant torque. To realize this condition, attach a string with a 2g weight to the top's cylinder, and run the wire over the air pulley, as shown the figure below.



Adjust the exhaust valve to change the air-jets flow until the top reaches a state closest as possible to the equilibrium. Remove the string from the top and measure the revolution periods. To keep the torque as constant as possible, never readjust the air pressure. To obtain a quasi-frictionless pulley, set the air flow of the pulley in such way the free pulley turns at very low speed. Try to keep the top as vertical as possible.
Use the provided reflective sensor and the matlab command MTCounterSave to measure the revolutions. For example, to count the top's revolutions for 80 s and save the time and revolutions number into a file called Top3Revs.Trial01.txt, type

```
MTCounterSave('Top3Revs.Trial01.txt',80);
```

Type help MTCounterSave for a more complete command description and usage.
REMEMBER TO REMOVE ALL THE SCOTCH TAPE FROM THE TOP’S RIM ONCE FINISHED.

3. Compare the two measured values of the moment of inertia I.

4. Compare the value of the angular acceleration $\ddot{\theta}_0$ obtained from the fit, with the value obtained from the definition of $\ddot{\theta}_0$ using the moment of inertia I calculated in point 1.

2.5 Second Laboratory Week

The purpose of this lab is to verify that the precession angular velocity Ω is independent of the angle ϕ and to study the Ω as a function of the top's center of mass.

If the center of the sliding mass m is placed at a distance h from the top's pivot, and h_0 is the position of the top's center of mass without m, the new center of mass will be located at (see Fig. 2.5)

$$h_{CM} = \frac{h_0 M + hm}{M + m} \tag{2.7}$$

It is important to notice that h_0 is negative because it is below the pivot O, which is the origin of the reference frame chosen to compute h_{CM}. With the addition of the mass m, equation (2.4) becomes

$$\Omega = \frac{(M+m)gh_{CM}}{I\omega}, \tag{2.8}$$

where we have neglected the small increase in the moment of inertia I due to the mass m. Inserting equation (2.7) into the equation (2.8), and after some algebra, we obtain

$$\Omega = \frac{Mg}{I\omega}h_0 + \frac{mg}{I\omega}h. \tag{2.9}$$

which relates the precession angular velocity to the sliding mass position h.

If we impose

$$h = h^* = -h_0\frac{M}{m}, \tag{2.10}$$

the angular velocity Ω of the precession goes to zero. If the mass m is placed at $h*$, h_{CM} is zero and the torque vanishes, and therefore the top does not precess.

It is important to notice that the spindle length is such that we can change the sign of h_{CM}.

2.5.1 Propaedeutic Problems.

1. The position of the top center of mass h_0 without the sliding mass m, is negative (below the pivot point). Provide a sketch depicting the direction of the angular velocity ω and the direction of the top precession.

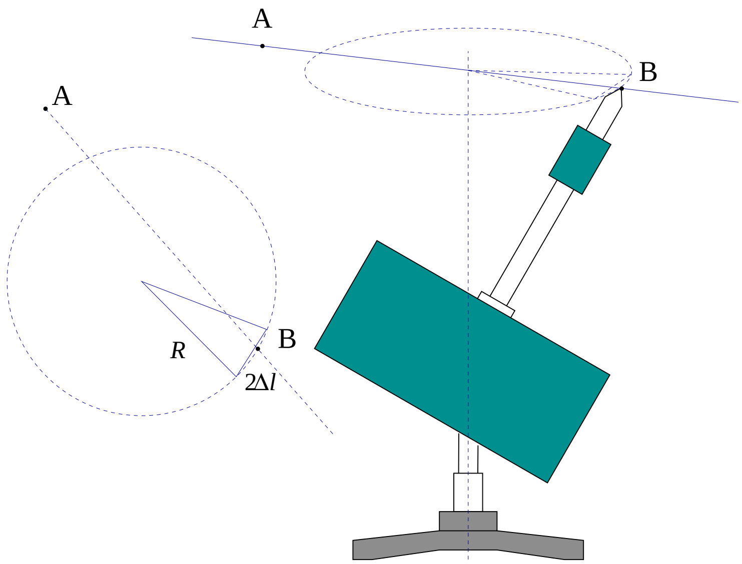
2. Calculate the period of precession T for a top spinning at 5Hz (5 revolutions per second, $\omega = 2\pi \times 5\text{rad/s}$) if $m = 0.2186\text{kg}$, $I = 4.66 \cdot 10^{-2}\text{kg m}^2$, and $h = h^* + 0.01\text{m}$.

3. Given the sliding mass $m = 0.2186\text{kg}$ with its outside diameter $D = 0.033\text{m}$, and inside diameter $d = 0.016\text{m}$, calculate the sliding mass moment of inertia I_m. Is the statement under equation (2.8) justified?

4. A linear fit to Ω versus h gives $\Omega = a + bh$. What are a and b in terms of m, g, I, ω and h^*? Supposing that ω is constant during

each measurement but different every time we change h, how can we rearrange equation 2.9 to still use a straight line as fitting function ?

5. The precession period T is measured with the sliding mass removed, and for a given value of the angular velocity ω. Write the equation that gives h_0 in terms of ω, T, M, I and g.



6. Using two points, **A** and **B,** to align the line of sight (see figure above), a careful student determines that the uncertainty of measuring where the spindle passes the pointer at **A** is $\Delta l = \pm 2$mm. If the radius of the precession orbit is $R = 50$mm, and the period is $T = 60$s, what uncertainty σ_T does this produce in the period T? What fraction is σ_T of the total period T?

2.5.2 Procedure (Precession Period Measurement)

Remember to follow the directives written in section 2.3.1 (Care and Use of the Experimental Apparatus) before starting the procedure. Use the reflective sensor and the program Tachometer available form the computer desktop to measure the revolution frequency of the top.

Setting the revolution frequency of the top at around 5Hz make the following measurements:

1. Without the sliding mass, demonstrate that top precession period T is independent from the angle ϕ for constant value of ω.

2. Using the previous measurements of the precession period T , of the angular revolution frequency ω, and of the moment of inertia I calculate $h^*$and its uncertainty . Place the sliding mass at h^* and confirm that the top does not precess.

3. Experimentally study the equation of the precession angular velocity Ω as a function to the sliding mass position h. Be sure that you measure the length of the sliding mass.

4. Compare the new measurement of I obtainable from step 3 with the two ones of the previous week.

5. Calculate the value of h^* obtainable from step 4 and compare it with the previous measurement.

6. REMEMBER TO CLOSE THE AIR SUPPLY OUTPUT ONCE FINISHED.