Ph 3 - INTRODUCTORY PHYSICS LABORATORY
– California Institute of Technology –

The Magneto-Mechanical Harmonic Oscillator

1 Introduction

The Harmonic Oscillator (sometimes called the Simple Harmonic Oscillator) plays a central role in modern physics and technology. For example, the mathematics describing simple harmonic motion provides the foundation for the development of wave mechanics and quantum mechanics, which you will see much more of in Ph2, Ph12, Ph125, as well as in many more advanced courses. Harmonic motion can describe the behavior of mechanical systems, electromagnetic systems, quantum mechanical systems, acoustic systems, and a broad range of other physical phenomena. Mechanical oscillators made from small plates of quartz crystal also form the foundation of timekeeping and frequency reference in electronic devices. Essentially every cell phone, timepiece, microwave transmitter, computer, and many other electronic devices contain quartz mechanical oscillators, and they are currently being manufactured at a rate of well over a billion units per year.

The focus of this lab is on understanding the Harmonic Oscillator, using a large-scale example where you can see rather directly how it responds to various stimuli. As you examine this very basic physical system, you should gain some intuition about how harmonic oscillators behave, and you should better understand how to connect the mathematics to the physics.

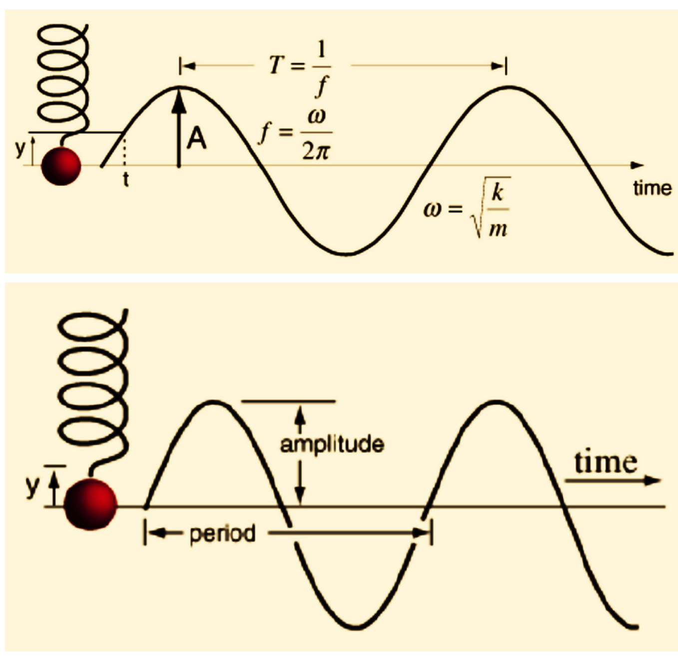


Figure 1. Basic terminology and notation for simple harmonic motion. (Image source: http://hyperphysics.phy-astr.gsu.edu/hbase/shm.html)

2 Simple Harmonic Motion - Theory

Before we look at the specific apparatus, let us first review the mathematics of simple harmonic motion, thereby defining our variables and examining how oscillators behave. We begin with the canonical example of a mass on a spring, as shown in Figure 1. One thing that makes this a *simple* harmonic oscillator is that we assume a purely linear, Hooke's-law spring constant, giving a restoring force $F = -kx$, where k is a constant. In the absence of any damping, the equation of motion for this system is

$$\begin{aligned} F &= ma = -kx \\ m\frac{d^2x}{dt^2} &= m\ddot{x} = -kx \\ m\ddot{x} + kx &= 0 \end{aligned}$$

Solutions to this equation are of the form

$$x(t) = C_1 \cos(\omega_0 t) + C_2 \sin(\omega_0 t) \tag{1}$$

where

$$\omega_0 = \sqrt{\frac{k}{m}} \tag{2}$$

while C_1 and C_2 are arbitrary constants. The constant $\nu_0 = \omega_0/2\pi$ is called the *resonant frequency* of the oscillator, measured in Hertz. (The constant ω_0 is also called the resonant frequency, more precisely the resonant *angular frequency*, this being measured in radians per second. Frequency measurements are usually reported in Hertz, while ω_0 is often more convenient for doing theory.) Note that the equation of motion is often written in the convenient form

$$\ddot{x} + \omega_0^2 x = 0 \tag{3}$$

The general theory of differential equations (not covered here!) tells us that Equation 1 is the full solution to this equation of motion. All we need to supply is the appropriate choice of C_1 and C_2. For example, if we know the initial position $x(t = 0)$ and velocity $\dot{x}(t = 0)$, then plugging in these *initial conditions* allows us to solve for C_1 and C_2, and from this we can predict the motion $x(t)$ for all future times.

We often use complex notation when talking about harmonic oscillators, for reasons described in Appendix 1 below. The (complex) solution becomes $x(t) = \tilde{A}e^{i\omega_0 t}$, where $\tilde{A}$ is a complex constant. The physical solutions are then either the real or imaginary parts of the complex solution, giving two arbitrary constants (the real and imaginary parts of $\tilde{A}$), and these constants are related to the C_1 and C_2 in Equation 1. The complex notation is often exceedingly useful because of its simplicity (dealing with $e^{i\omega t}$ is less cumbersome compared with sines and cosines), and we will see this more below. But one should note that this is a bit of a shorthand notation, which can lead to problems. Just remember that in the final analysis it is the real, physical, solutions that describe the system.

2.1 Energetics

Consider the general oscillator solution in Equation 1, which we can also write in the form $x(t) = A\sin(\omega_0 t + \delta)$. The kinetic energy of the mass motion is

$$\begin{aligned} E_{kinetic} &= \frac{1}{2}mv^2 = \frac{1}{2}m\dot{x}^2 \\ &= \frac{1}{2}m\omega_0^2 A^2 \cos^2(\omega_0 t + \delta) \end{aligned}$$

and the potential energy stored in the spring is

$$\begin{aligned} E_{potential} &= \int_0^x F(x')dx' = \int_0^x kx'dx' \\ &= \frac{1}{2}kx^2 \\ &= \frac{1}{2}kA^2\sin^2(\omega_0 t + \delta) \\ &= \frac{1}{2}m\omega_0^2 A^2 \sin^2(\omega_0 t + \delta) \end{aligned}$$

where we used Equation 2 to obtain the last expression.

From these we see that the total energy

$$\begin{aligned} E_{total} &= E_{kinetic} + E_{potential} \\ &= \frac{1}{2}m\omega_0^2 A^2 \end{aligned} \tag{4}$$

is independent of time. As the mass oscillates, energy sloshes back and forth between kinetic energy and potential energy.

2.2 The Damped Harmonic Oscillator

We add damping to our simple harmonic oscillator using the damping force $F = -\gamma\dot{x}$, where γ is a constant. The equation of motion ($F = ma$) then becomes

$$\begin{aligned} m\ddot{x} &= -kx - \gamma\dot{x} \\ m\ddot{x} + \gamma\dot{x} + kx &= 0 \\ \ddot{x} + \Gamma\dot{x} + \omega_0^2 x &= 0 \end{aligned} \tag{5}$$

where $\Gamma = \gamma/m$ and again we use $\omega_0^2 = k/m$.

To solve this equation, we try a solution of the form $x = Ae^{i\omega t}$, where A is a complex constant and ω is a real constant. Plugging this in gives

$$\left(-\omega^2 + i\omega\Gamma + \omega_0^2\right) x = 0$$

Assuming $x \neq 0$, we have

$$\omega^2 - i\omega\Gamma - \omega_0^2 = 0$$

and solving this quadratic equation gives

$$\omega = \frac{i\Gamma}{2} \pm \sqrt{\omega_0^2 - \frac{\Gamma^2}{4}} \tag{6}$$

Our final real solution is then $x = \mathrm{Re}[Ae^{i\omega t}]$, where A is an arbitrary complex constant and ω is given by the previous expression. At this point we will limit ourselves to the regime $\Gamma^2 < 4\omega_0^2$, which is called the *underdamped* case. In this case the test mass oscillates away, as in the zero damping case, but the oscillations slowly decrease in amplitude with time, eventually settling to $x = 0$. (In the opposite case, $\Gamma^2 > 4\omega_0^2$, called the *overdamped* case, the test mass just damps down without oscillating, which is not terribly interesting. Most science and technology applications focus on the underdamped case as well.)

To simplify the notation a bit, we define the damped oscillator frequency ω_d, where

$$\omega_d^2 = \omega_0^2 - \frac{\Gamma^2}{4}$$

and we further define the decay time T

$$T = \frac{2}{\Gamma}$$

so we can write the full (still complex) solution

$$x(t) = Ae^{-t/T}e^{\pm i\omega_d t}$$

where A is a complex constant while T and ω_r are both real numbers. Converting to only real quantities, the full solution becomes

$$x(t) = C_1 e^{-t/T} \cos(\omega_d t) + C_2 e^{-t/T} \sin(\omega_d t) \tag{7}$$

where C_1 and C_2 are real constants. And again, these constants are determined by the initial conditions of our oscillator – the position and velocity at $t = 0$. This solution can also be written

$$x(t) = C_3 e^{-t/T} \cos(\omega_d t + \delta)$$

where now C_3 and δ are real constants. One can expand $\cos(\omega_d t + \delta)$ to find the relationship between (C_1, C_2) and (C_3, δ), so either set of constants can be derived from the other.

To summarize, the underdamped oscillator looks a lot like the undamped oscillator, except that the amplitude of the oscillations decay exponentially with a time constant T.

2.3 The Quality Factor

We also define a *quality factor* Q for the oscillator,

$$\begin{aligned} Q &= \pi \frac{\text{Decay Time}}{\text{Resonant Period}} \\ &= \pi \frac{T}{2\pi/\omega_d} \\ Q &= \frac{\omega_d T}{2} = \frac{\omega_d}{\Gamma} \end{aligned}$$

Note that Q is a dimensionless number, roughly equal to the number oscillation cycles that occur before the amplitude decays away. (More precisely, the amplitude decays to $e^{-\pi}$ times its original value after Q cycles.) This is often written as

$$Q = 2\pi \frac{\text{Energy Stored}}{\text{Energy Loss per Cycle}}$$

and the reader can verify that these two definitions are the same.

In this lab you will be working with an oscillator with $Q > 100$. In this case we can expand ω_d for small Γ, giving

$$\begin{aligned} \omega_d &= \left(\omega_0^2 - \frac{\Gamma^2}{4}\right)^{1/2} \\ &= \omega_0 \left(1 - \frac{\Gamma^2}{4\omega_0^2}\right)^{1/2} \\ &\approx \omega_0 \left(1 - \frac{1}{8Q^2}\right) \end{aligned}$$

What this means is that ω_d will equal ω_0 to better than a part in 10^4, and such a small difference will be negligible in this lab. We will therefore assume $\omega_d = \omega_0$ in the discussion that follows, and this greatly simplifies the math. The quality factor Q can then be written

$$\begin{aligned} Q &\approx \frac{\omega_0 \tau}{2} = \frac{\omega_0}{\Gamma} = \frac{2\pi\nu_0}{\Gamma} \\ &\approx \pi \nu_0 \tau \end{aligned}$$

Note that the quality factor Q and the damping constant Γ are related. For an underdamped oscillator, we can talk about the damping constant Γ, or the decay time T, or the quality factor Q. They all refer to basically the same thing, that the oscillation amplitude decays away with time.

2.4 The Driven Harmonic Oscillator

Finally, we take the last step and drive our oscillator with a sinusoidal force applied at some angular frequency ω. This further complicates things, but it is important, so keep reading. Imagine you are in a playground applying a sinusoidal force to a swing (not so easy to do a sinusoidal force, but try). This helps give you some intuition about the math.

The equation of motion ($F = ma$) becomes

$$\ddot{x} + \Gamma\dot{x} + \omega_0^2 x = (F_{applied}/m)\, e^{i\omega t} = F_m e^{i\omega t} \tag{8}$$

Here we have defined a normalized force $F_m = F_{applied}/m$, where $F_{applied}$ has the actual dimensions of a force. Again we try a solution of the form $x(t) = Ae^{i\omega t}$, and plugging this in gives

$$\begin{aligned} \left(-\omega^2 + i\omega\Gamma + \omega_0^2\right) Ae^{i\omega t} &= F_m e^{i\omega t} \\ A &= \frac{F_m}{-\omega^2 + i\omega\Gamma + \omega_0^2} \\ &= \frac{F_m}{(\omega_0^2 - \omega^2) + i\omega\Gamma} \end{aligned} \tag{9}$$

This is sometimes called the *response function* of the oscillator. If you drive it with some force $F_{applied}$ at some frequency ω, the resulting amplitude of the oscillations is given by $|A|$. You can see from this expression that if you drive the oscillator near its resonant frequency ($\omega \approx \omega_0$), then the oscillation amplitude will be high. If you drive it far away from resonance, the amplitude will be lower. A playground swing behaves that way also.

2.5 The Full Solution, and the Steady-State Solution

Just for completeness, we can write down the full, real solution to Equation 8 (although we will not *prove* here that this is the unique solution; see Ma 2a for more on uniqueness theorems):

$$x(t) = \mathrm{Re}\left[Ae^{-t/T}e^{i\omega_d t} + \frac{F_m}{(\omega_0^2 - \omega^2) + i\omega\Gamma}e^{i\omega t}\right]$$

where $\mathrm{Re}[Fe^{i\omega t}]$ is the applied sinusoidal force and A is a complex constant that depends on the initial conditions. This is complicated, but you can see that for $t \gg T$ the initial conditions do not matter anymore. When $t \gg T$ we are left with the *steady-state solution*, which is simpler

$$x(t) = \mathrm{Re}\left[\frac{F_m}{(\omega_0^2 - \omega^2) + i\omega\Gamma}e^{i\omega t}\right]$$

Note that in steady-state $x(t)$ oscillates at the drive frequency ω (as seen by the $e^{i\omega t}$ term), which is generally not equal to the resonant frequency ω_0.

2.6 Some Simplifications

Now let us focus on the steady-state behavior of an underdamped oscillator. When the drive frequency is near ω_0, we can write $\omega = \omega_0 + \Delta\omega$ and expand for small $\Delta\omega$, giving

$$\begin{aligned} \omega_0^2 - \omega^2 &= (\omega_0 + \omega)(\omega_0 - \omega) \\ &\approx -2\omega_0\Delta\omega \end{aligned}$$

The complex constant A includes both amplitude and phase information. Focusing on the amplitude, we multiply A by its complex conjugate to get

$$\begin{aligned} \left|A^2\right| &= \left|F_m^2\right| \frac{1}{(\omega_0^2 - \omega^2) + i\omega\Gamma}\frac{1}{(\omega_0^2 - \omega^2) - i\omega\Gamma} \\ \frac{\left|A^2\right|}{\left|F_m^2\right|} &\approx \frac{1}{[-2\omega_0\Delta\omega + i\omega\Gamma]}\frac{1}{[-2\omega_0\Delta\omega - i\omega\Gamma]} \\ &\approx \frac{1}{4\omega_0^2\Delta\omega^2 + \omega_0^2\Gamma^2} \\ &\approx \frac{1}{4\omega_0^2}\frac{1}{\Delta\omega^2 + \Gamma^2/4} \end{aligned}$$

which is a simple Lorentzian function of $\Delta\omega$. The amplitude of the oscillations then becomes

$$|x| = \frac{1}{\sqrt{\Delta\omega^2 + \Gamma^2/4}}\frac{|F_m|}{2\omega_0} \tag{10}$$

When you do your data analysis in the lab, it is useful to rearrange this expression to yield the equivalent formula

$$|x| = \frac{A_1\,(\nu_0/2Q)}{\sqrt{(\nu-\nu_0)^2 + (\nu_0/2Q)^2}} \tag{11}$$

where A_1 is a constant.

The amplitude peaks when the oscillator is driven at its resonant frequency ($\omega = \omega_0$, $\Delta\omega = 0$), giving

$$|x|_{\text{on resonance}} = \frac{Q}{\omega_0^2}\,|F_m| = \frac{Q}{m\omega_0^2} F_{applied}$$

From this we see that the amplitude of a driven oscillator is proportional to Q when driven on resonance. With a very high quality factor, only a small driving force is needed to produce a large oscillation amplitude. Again, you know this from your playground experience.

If we drive the oscillator at very low frequencies, then $\dot{x}$ will be low also (since $\dot{x}$ is proportional to ω), and we can therefore ignore the damping term. In this case the amplitude of the oscillator becomes

$$\begin{aligned} A &= \frac{F}{(\omega_0^2 - \omega^2) + i\omega\Gamma} \\ |x|_{\text{static}} &\approx \frac{1}{\omega_0^2}\,|F| = \frac{1}{m\omega_0^2} F_{applied} \end{aligned}$$

This low-ω limit essentially gives us the static response of the oscillator. We can also get this by going all the way back to the beginning of our discussion. Hooke's law gives a restoring force $-kx$. Setting this equal to the applied force gives $F_{applied} = kx_{static}$, so $x_{static} = F_{applied}/k = F_{applied}/m\omega_0^2$.

Comparing $|x|_{\text{static}}$ and $|x|_{\text{on resonance}}$, we see that the resonant response is Q times larger than the static response. This is just what happens when you push on a swing; a small static force gives a small displacement, but a small force applied at the resonance frequency can give a large oscillation amplitude. And there you have it - playground physics in all its mathematical glory. (What is Q for a swing? Same as above – roughly the number of oscillations before the amplitude decays away.)

2.7 Phase Information

It is also instructive to look at the phase of the response x to the applied force F. At low frequencies x is proportional to F, meaning that the displacement is in phase with the drive. This should fit your intuition from the swing analogy as well. If you apply a steady force pushing a swing to the left, it moves to the left, and vice versa; thus the displacement is in phase with the applied force.

Far above the resonance frequency, the situation changes. Going back to our solution above, we have the complex amplitude

$$A = \frac{F_m}{(\omega_0^2 - \omega^2) + i\omega\Gamma} \tag{12}$$

When we are far above resonance ($\Delta\omega \gg \Gamma$ and $\omega^2 \gg \omega_0^2$), this becomes

$$A \approx -\frac{F_m}{\omega^2} = -\frac{F_{applied}}{m\omega^2}$$

and we see that the response is 180 degrees out of phase with the drive. When we are pushing to the left, the position of the mass is to the right, and vice versa.

This is exactly what happens when you apply a sinusoidal force to a free mass. Newton's law is simply (with no restoring force and no damping)

$$m\ddot{x} = F_{applied}$$

This is simple enough we can drop the complex analysis, and let $F_{applied} = F_{app,0}\cos(\omega t)$. Assuming a solution $x(t) = A\cos(\omega t)$ gives

$$\begin{aligned} -mA\omega^2 \cos(\omega t) &= F_{app,0}\cos(\omega t) \\ A &= -\frac{F_{app,0}}{m\omega^2} \end{aligned}$$

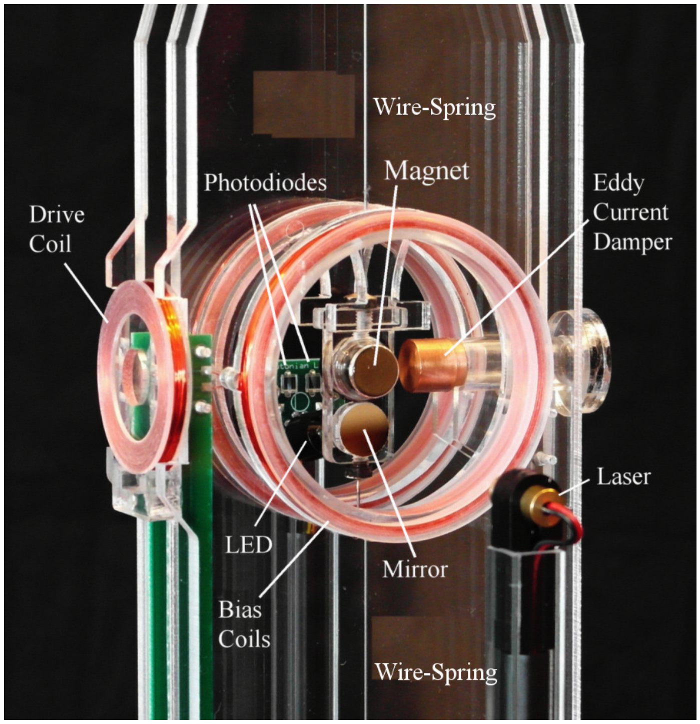


Figure 2. Components of the Magneto-Mechanical Harmonic Oscillator.

which again shows that the displacement is 180 degrees out of phase with the applied force.

When the applied force is on resonance, the solution in Equation 12 becomes

$$A = \frac{F_{app,0}}{im\omega\Gamma}$$

so the response is 90 degrees out of phase with the drive. Thus we see the transition – from in-phase at low drive frequencies (the static response), to 90 degrees out of phase on resonance, to 180 degrees out of phase at high drive frequencies (the free-mass response).

2.8 The Torsional Oscillator

In the lab you will be working with a *torsional oscillator*, which is a special form of a simple harmonic oscillator. In a torsional oscillator, linear motion is replaced by angular motion. So the displacement x is replaced by the angular displacement θ. Newton's law $F = ma$ is replaced by its torque version

$$\tau = I\ddot{\omega}$$

where $\ddot{\omega} = d^2\omega/dt^2$ is the angular acceleration, and I is the mass moment of inertia. The restoring force becomes a restoring torque

$$\tau = -\kappa\theta$$

The math all follows exactly the same as it did above, giving us a resonance frequency ω_0, a decay time T, a quality factor Q, etc. The resonant frequency in the torsional case, for example, becomes

$$\omega_0 = \sqrt{\frac{\kappa}{I}} \tag{13}$$

3 Laboratory Exercises – Week One

What follows are step-by-step instructions that will walk you through this lab. Each paragraph describes a task or two, and you should complete the task(s) in one paragraph before moving on to the next.

- Begin your laboratory session by removing all the cables attached to the Magnet-Mechanical Harmonic Oscillator (MMHO) chassis. Then turn it on using Laser switch on the chassis. You should see a red laser turn on, and also a bright LED. At the center of the MMHO tower you can see a cylindrical magnet, 0.75 inches long and 0.5 inches in diameter, that is magnetized along the axis of the cylinder. Just below the magnet you can see two small mirrors, one on each side of the plastic plate holding the magnet. The laser beam should be reflecting off one mirror and hitting a plastic ruler about 75 cm away. If not, adjust the tripod so the laser hits the ruler. Observe that the magnet assembly is supported by two steel wire-springs, above and below the magnet. These wires provide the (twisting) restoring torque $\tau = -\kappa\theta$ in the torsional oscillator.

- Next turn on the Waveform Generator and your Oscilloscope. Look at the Ch 1 output from the waveform generator on the oscilloscope. Trigger the scope so you see a nice stationary sine wave on the oscilloscope. In general, you should always view any signals on the oscilloscope if you can. Otherwise you are operating blind, and this practice typically ends up wasting more time than the time it takes to check your signals as you go.

- Now adjust the waveform generator to produce a sine wave at 40 Hz, with an amplitude of 5 volts (V_{PP} on the function generator). View this signal on the oscilloscope also. When that looks good, send the signal to the *Drive Coil IN* port instead of to the oscilloscope. You should see the laser spot on the ruler turn into a short streak. The Drive Coil is the small coil located at the back of the MMHO tower. With your sine-wave input, this coil generates an oscillating magnetic field that is perpendicular to the magnet, and this field exerts a torque

  $$\tau = \vec{\mu} \times \vec{B} \approx \mu B \approx \mu B_0 \cos \omega t$$

  on the magnet (the approximation being valid for small θ), driving the torsional oscillator. Here μ is the magnetic moment of the magnet and B is the magnetic field from the drive coil, at the position of the magnet, and B_0 is a constant. The oscillations are fast enough (around 40 Hz) that the sweeping laser spot looks like a streak. (Safety note; If the streak is ever longer than the ruler, turn the drive amplitude down a bit. At sufficiently high amplitudes it may be possible to damage the MMHO.)

- Next find the eddy current damper, which is a copper cylinder at the end of a short plastic tube. This may already be installed in the MMHO tower, or if not it should be on the lab bench somewhere nearby. Place it into the front of the MMHO tower, so the copper cylinder is near the magnet. When the magnet oscillates back and forth, it produces changing magnetic fields inside the copper. These changing fields induce currents in the bulk of the copper, called eddy currents. The currents cause Ohmic heating inside the copper that dissipate energy and slow the motion of the magnet. (Actually the eddy currents produce magnetic fields that drive the oscillator to lower amplitudes. You can think about the magnetic torques here, or you can think about the energy dissipation in the copper, the energy being supplied by the oscillating magnet. Both give the same result – damping of the oscillator.) One nice feature of eddy current damping is that it is accurately described by a simple damping torque $\tau = -\gamma\dot{\omega}$, where γ is a constant (analogous to the damping force $F = -\gamma\dot{x}$ described in the theory section).

- With the eddy current damper in place, turn up the drive amplitude and adjust the frequency of the drive. You should see the oscillation amplitude reach a peak when you are at the resonance frequency ν_0 of the oscillator. Determine ν_0 to the nearest 0.1 Hz and record this in your notebook.

- With the drive frequency near ν_0, press the Output button on the function generator to turn off the drive. The oscillation amplitude decays to zero. Turn the drive back on, and the amplitude goes back up. Nothing surprising there. Now remove the eddy current damper and repeat this experiment. With less damping, you will see the oscillator amplitude go higher, and the decay to zero will take longer. Makes sense. (Again, keep the laser streak shorter than the ruler at all times.)

- Now turn the drive off, let the oscillator die down, set the drive frequency to $(\nu_0 - 1)$ Hz, set the drive amplitude to its maximum, remove the eddy current damper, and turn the drive back on again. This time the behavior is more complicated. You should see a beating between two frequencies – the natural resonant frequency of the oscillator and the drive frequency. In this case the difference is 1 Hz, so this is the beat frequency. Try the same experiment with the eddy current damper left in. You should see the same basic behavior, but it takes less time to reach the steady state. In a nutshell, at early times you are seeing the full solution to the driven harmonic oscillator, which can be complicated. But after a while the oscillator settles down into its steady-state solution, as described in the theory section above.

- Next put the eddy-current damper in and set the drive frequency to ν_0. Using another BNC cable, connect the *Photodiodes OUT* port to channel 1 of the oscilloscope. On the right side of the MMHO column you can see a bright LED shining into one of the small mirrors below the test magnet. A spot of light from this LED is reflected onto two small rectangular photodiodes. The difference signal V_{diff} from these two photodiodes gives the *Photodiodes OUT* signal. If the oscillator has zero amplitude, so $\theta = 0$, then both photodiodes see the same amount of light, so the difference signal is $V_{diff} = 0$. If $\theta > 0$, then one photodiode sees more light and the difference signal is positive. If $\theta < 0$, then the other photodiode sees more light and the difference signal is negative. For small θ, the *Photodiodes OUT* signal is proportional to the oscillator angle, so $V_{diff} = C_{diff}\theta$, where C_{diff} is a constant.

- Turn up the drive amplitude and watch what happens to the photodiode signal. At very high amplitudes, the signal goes down because the reflected LED spot starts missing both photodiodes. Observe this behavior on the oscilloscope. For small θ, the *Photodiodes OUT* signal is proportional to θ, so an oscillating test mass gives a sinusoidal signal on the oscilloscope. But you can see that the waveform becomes distorted at high oscillation amplitudes, even though the test mass is still exhibiting simple sinusoidal oscillations.

- Next set the drive amplitude to 2 volts and use a BNC Tee to send the drive signal to both the *Drive Coil IN* and to channel 2 of the oscilloscope. Watch both oscilloscope traces together as you change the drive frequency. For best results, trigger on channel 2 on the oscilloscope. When $\nu \ll \nu_0$, you should see the oscillations in phase with the drive. And when $\nu \gg \nu_0$ you should see the two signals 180 degrees out of phase. And when $\nu = \nu_0$, you should see the two signals out of phase by 90 degrees. If you set the oscilloscope to XY display mode, you can see the phase relationship between the signals more clearly, and you can tweak ν to see when the phase difference is nearly exactly 90 degrees, which should be at ν_0.

- Next it's time to get quantitative. In this exercise you will measure the oscillation amplitude as a function of the drive frequency. Move the tripod so it is close to 75 cm from the center of the MMHO tower; measure this distance to the nearest cm and record it. Put the eddy current damper back in (make sure it is all the way in), set the drive frequency to ν_0, and adjust the drive amplitude so the laser streak is just a bit shorter than the ruler. Tweak the frequency and make sure the laser streak always stays on the ruler (not beyond). Adjust the drive amplitude to make this happen. Now adjust the tripod so the laser streak strikes close to the top edge of the ruler, in the millimeter divisions. Make sure the ruler is perpendicular to the laser beam (when the beam has zero amplitude), and make sure the laser streak is nicely parallel to the edge of the ruler. A little care setting this up now gives you better data later.

- Next measure the oscillation amplitude as a function of drive frequency. First turn the drive off and record the position of the laser spot. With the drive on, you then only need to measure one end of the laser streak, which saves time over measuring both ends. Record a series of measurements where you: 1) change the drive frequency; 2) wait for the oscillation amplitude to stabilize (a few seconds); and 3) record the position of the end of the laser streak. Scan the drive frequency both above and below ν_0. You will not want even divisions in drive frequency – that would give you too few data points near resonance, and too many data points far away from resonance. A good rule-of-thumb is to adjust the drive frequency so the oscillation amplitude changes by about 20 percent each step (roughly; just eyeball it). In addition, take a few extra points when you are right near ν_0. Be careful not to disturb anything during your measurements. If you bump the ruler, for example, then you just have to start over again. So carefully change the drive amplitude and nothing else between data points. If you become unhappy with your measurements for any reason, don't hesitate to abort and start over. When you have things going smoothly, it does not take long to collect a good set of data. Twenty points is plenty. At the end, turn off the drive and measure the zero-amplitude position again. (If it moved, this gives you some indication of how stable the system was, and how accurate your measurements might be. Things do drift with time, so don't worry if the zero-amplitude point moved a millimeter. But if it moved a lot, then your data were probably corrupted somehow.)

- To analyze your data, subtract the zero-amplitude position and plot the oscillatory amplitude as a function of drive frequency, $x(\nu)$. Do your analysis in a Mathematica notebook, as demonstrated in the Appendix below. (If you are proficient with Mathematica already, do not consult the Appendix unless you get stuck.) The $x(\nu)$ data should be described by the functional form in Equation 11. Before doing a fit to the data, try a quick "chi-bye-eye" estimate – draw a theory curve on the same plot as your data, and fiddle with the parameters to make the curve go through the data. From your fit, extract the resonance frequency ν_0 and the mechanical Q of the oscillator, and write these in your notebook (with standard errors).

4 Laboratory Exercises – Week Two

4.1 Adding a Magnetic Restoring Force

For the next part of the lab you will apply a magnetic restoring torque, adding this to the restoring torque from the supporting wire-springs. If we apply a constant magnetic field of strength B_0 in the $\theta = 0$ direction, then the magnetic torque on the test mass is

$$\begin{aligned} \tau_{magnetic} &= \vec{\mu} \times \vec{B} \\ &= -\mu B_0 \sin\theta \end{aligned}$$

where θ is the angular position of the test mass. Using the small-angle approximation $\sin\theta \approx \theta$, we can write the total restoring torque

$$\begin{aligned} \tau &= -\kappa_0\theta - \mu B_0\theta \\ &= -(\kappa_0 + \mu B_0)\theta \\ &= -\kappa_{total}\theta \end{aligned}$$

where μ is the magnetic moment of the test mass and κ_0 is the spring constant provided by the support wires. With a nonzero B_0, the resonant frequency of the oscillator becomes

$$\omega_0 = \sqrt{\frac{\kappa_{total}}{I}}$$

where I is the moment of inertia of the oscillator. For small B_0 the frequency change is

$$\begin{aligned} \Delta\omega_0 &\approx \frac{1}{2}\left(\frac{\kappa_{total}}{I}\right)^{-1/2}\frac{\Delta\kappa_{total}}{I} \\ \frac{\Delta\omega_0}{\omega_0} &\approx \frac{1}{2}\frac{\Delta\kappa_{total}}{\kappa_{total}} \\ &\approx \frac{1}{2}\frac{\mu B_0}{\kappa_0} \end{aligned}$$

For analyzing your data, it is convenient to rewrite this as

$$\nu \approx \nu_0 + \frac{\mu\nu_0\beta}{2\kappa_0} i_{bias}$$

where the applied magnetic field is proportional to the current in the bias coils, give by $B_0 = \beta i_{bias}$.

This change in the resonant frequency of the oscillator allows you to measure the magnetic moment of the test mass. You first calculate I (see below), then combine this with the known resonant frequency and I is the mass moment of inertia $\omega_0 = 2\pi\nu_0$ to determine κ_0. Then you measure ν as a function of the applied current i_{bias} to determine μ. As before, we proceed with a step-by-step list of procedures that will guide you through the lab.

- Begin by installing the eddy-current damper into the MMHO tower as usual and disconnecting all cables from the MMHO chassis. Then connect the *Clock Drive OUT* signal to the *Drive Coil IN* port using a BNC cable. Turn up the *Feedback Gain* knob and you will see the oscillator amplitude go up. To see what this is doing, look at the *Photodiodes OUT* signal using the oscilloscope. This should be familiar from your previous lab session. Attach a BNC Tee to the *Clock Drive OUT* port so you can observe this signal simultaneously on the oscilloscope (while it remains connected to the *Drive Coil IN* port to drive the oscillator). The electronics inside the MMHO chassis first takes the *Photodiodes OUT* signal and compares it with zero to produce a square wave signal. The electronics then takes the derivative of this square wave, which is essentially the pulsed output you see with the *Clock Drive OUT*. The amplitude of this signal is set by the *Feedback Gain* knob, and you can see this on the oscilloscope. When you send this signal to the *Drive Coil IN*, then every time the oscillator goes through $\theta = 0$, it gets an impulse, which is a short drive torque. If you think about, you will see that these impulses alternate in sign, so each kick tends to increase the amplitude of the oscillator. The amplitude increases until the impulses are balanced by the internal damping of the oscillator.

- Note that this system uses feedback to sense the position θ of the oscillator, then uses that information to provide a drive force. You typically do this when you push a playground swing as well. In the MMHO we call this Clock Mode, because this is essentially how all clocks work – feedback keeps the oscillator going, and one counts pulses (in essence) to keep time. (In a purely mechanical clock, like a pendulum clock, the feedback is supplied by a clever mechanism called the *escapement*. You can look this up if you are interested.) Note that because the drive is provided by the motion of the oscillator itself, it should run at its resonant frequency ν_0.
- Use the measure feature on you oscilloscope to measure the frequency of the *Photodiodes OUT* signal, giving you ν_0. This works okay, but is not terribly accurate. You can do better using the function generator, which is something of a precision timepiece. Send a square wave from the function generator to the oscilloscope, viewing this in addition to the *Clock Drive OUT* signal. Trigger on the square wave, and you will see the *Clock Drive OUT* signal drift on the oscilloscope. Adjust the function generator frequency until the drift stops; then the two frequencies will be equal. The accuracy of the measurement is mainly limited by how long you are willing to watch the signal drift on the oscilloscope. Note that if you line up a sharp edge of the *Clock Drive OUT* with a sharp edge of the square wave, and expand the timebase on the oscilloscope, you can see very small phase shifts quickly, allowing you to measure the oscillator frequency with high accuracy. Measuring ν_0 to a milliHertz accuracy only takes a few minutes this way. See how well you can measure ν_0, and write this in your notebook (with your rough estimate of the measurement accuracy).
- The resonant frequency also changes with temperature. Check this out by blowing gently into the MMHO tower to heat the wire-springs a small amount. You should find that your precise measurements of ν_0 are probably limited by thermal drifts that you cannot easily control or monitor with this apparatus.
- Just for fun, change the function generator signal to a pulsed signal (press the Pulse button), set the amplitude to 5 volts, and the duty cycle to 10 percent, and feed this signal into the *Laser Strobe IN* port. As the name implies, this uses the input pulses to strobe the laser – on when the signal is high, off when the signal is low. See what happens when you vary the frequency. Try near ν_0, and multiples of ν_0.
- Next, use your measurement of ν_0 to determine the spring constant κ_0. For this you will need the mass moment of inertia I and Equation 13. If you do a search for "moment of inertia formula cylinder", you will soon find
$$I = \frac{1}{12} m \left(3R^2 + L^2\right)$$
for our situation, where m is the cylinder mass (the cylinder being the test-mass magnet), R is the radius, and L is the length. The magnet has $2R = 0.5$ inches, $L = 0.75$ inches, and $m = 18$ grams. Add 10 percent to I for the added contribution from the small mirrors and the plastic mount (an estimate here is sufficient), and record I in your notebook (with units). Calculate κ_0 and write this down also (interestingly, κ_0 has the units of Joules).
- Next connect a DC power supply to the *Bias Coils IN* port, which sends current to the pair of large coils around the MMHO tower. Keep the applied current below 1 Amp, to avoid burning up the coils. This pair of coils generates a magnetic field in the $\theta = 0$ direction. From the coil geometry, the calculated magnetic field at the test mass is $B_0 = \beta i_{bias}$, where $\beta = 0.0051$ Tesla/Amp (to an accuracy of about 10 percent). Measure ν_0 as a function of i_{bias}, and use this to determine the magnetic moment of the test mass, as described in the theory section above. Again, analyze your data in a Mathematica notebook, and consult the Appendix below if you get stuck.
- Rare-earth magnets like the one in the MMHO typically have a magnetic moment per unit volume of roughly 10^6 A/m (that is, Amps/meter). Multiply this by the magnet volume to get a rough estimate for μ. (Note that one Joule/Tesla is the same as one A-m^2; electromagnetic units are funny that way.)
- Now assume that the magnet is made entirely of neodymium. What is the effective magnetic moment per neodymium atom? How many Bohr magnetons is this? What a permanent magnet does is essentially line up the electron spins (one Bohr magneton per electron) in the material as much as possible, and each electron magnetic moment contributes to the total magnetic moment of the magnet (that is an oversimplification, but not too far off). Simple chemistry tells us that electrons like to be paired with opposing spins, and this is the main limitation to how strong you can make a permanent magnet.

5 Appendix I: Using Complex Functions to Solve Real Equations

In case this is not familiar to you already, physicists and engineers often use complex functions to solve real equations, with the understanding that you take the real part at the end. Why does this work? And why do we even do this? We can demonstrate with the simple harmonic oscillator. Start with the equation of motion $\ddot{x} + \omega_0^2 x = 0$, and let us solve this using a complex function: $x = \alpha + i\beta$, where $\alpha(t)$ and $\beta(t)$ are real functions. If you plug this in, you will see that $\ddot{x} + \omega_0^2 x = 0$ becomes

$$\begin{aligned} \left(\ddot{\alpha} + i\ddot{\beta}\right) + \omega_0^2 \left(\alpha + i\beta\right) &= 0 \\ \left(\ddot{\alpha} + \omega_0^2 \alpha\right) + i\left(\ddot{\beta} + \omega_0^2 \beta\right) &= 0 \end{aligned} \tag{14}$$

Since a complex number equals zero only if both the real and imaginary parts equal zero, we see that $\ddot{x} + \omega_0^2 x = 0$ implies that both $\ddot{\alpha} + \omega_0^2 \alpha = 0$ and $\ddot{\beta} + \omega_0^2 \beta = 0$. In other words, both the real and imaginary parts of $x(t)$ satisfy the original equation.

So we have a procedure: try using a complex function to solve the original equation. If this works, then taking the real part of the solution gives a real function that also solves the same differential equation. (If in doubt, then verify directly that the real part solves the equation.)

Why do we go to the trouble of using complex functions to solve a real equation? Because differential equations are often easier to solve when we assume complex functions (seems counterintuitive, but it's true). The function $e^{i\omega t}$ is a simple exponential, and the derivative of an exponential is another exponential – that makes things simple. In contrast, cosines and sines are more difficult to work with.

In the case of the simple harmonic oscillator, the solution $x(t) = Ae^{i\omega t}$, where A is complex, has a natural interpretation. The length and angle of the A vector (in the complex plane) give the amplitude and phase of the oscillations.

You should note, however, that this only works for linear equations. If our equation were $\ddot{x} + \omega_0^2 x + \gamma x^2 = 0$, for example, then using complex functions would not have the same benefits. In fact, there is no simple solution to this equation, complex or otherwise. This equation describes a nonlinear oscillator, and nonlinear oscillators exhibit a fascinating dynamics with interesting behaviors that people still study to this day.

Fitting MMHO data, week 1: Amplitude vs Drive frequency

For this measurement, you probably recorded the endpoint of the laser streak on the ruler as a function of drive frequency, and you recorded the zero point (the ruler position of the laser spot with the oscillator at rest).

Now you want to fit those data to extract the resonant frequency and the oscillator Q. Theory says the data should fit the functional form in equation 11, so you will want to convert your ruler positions to oscillator amplitude. You could do this by hand using a calculator, but it is better to let *Mathematica* do the calculations for you.

For example, first enter your numbers into a *Mathematica* list like so (only showing two points here):

```
In[157]:= data1 = {{freq1, pos1}, {freq2, pos2}}

Out[157]= {{freq1, pos1}, {freq2, pos2}}
```

Here is one way to convert those ruler endpoint positions to amplitude. First extract the ruler data by itself and subtract the zero point:

```
zeropos = 20; (* this is the zero point position on the ruler *)
(* note the semicolon suppresses output for this line *)
amplitude =
 data1[[All, 2]] - zeropos (* the oscillator amplitude in millimeters *)

Out[159]= {-20 + pos1, -20 + pos2}
```

(Ideally we would convert the length of the streak to oscillation amplitude in radians, but the tangent correction factor is small, so leaving it in millimeters is okay.)
Then convert everything back to a data array in (x,y) format:

```
In[160]:= frequency = data1[[All, 1]] ;
data = Table[{frequency[[i]], amplitude[[i]]}, {i, 1, Length[amplitude]}]

Out[161]= {{freq1, -20 + pos1}, {freq2, -20 + pos2}}
```

Your final data set is then a set of points in the form {frequency, oscillator amplitude}
To proceed with this example, we will use some simulated data:

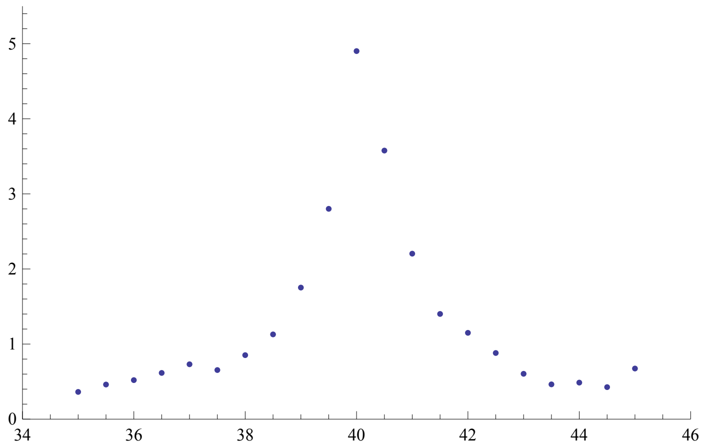
```
In[162]:= amp = 5;  (* amplitude *)
freq0 = 40.1; (* the center frequency *)
qq = 50; (* the oscillator Q *)
data =
 Table[{x, ((amp * freq0) / (2 * qq)) / Sqrt[((x - freq0)^2 + ((freq0 / (2 * qq))^2))] +
    0.1 * RandomVariate[NormalDistribution[]]}, {x, 35, 45, 0.5}]

Out[165]= {{35., 0.362544}, {35.5, 0.460881}, {36., 0.51963}, {36.5, 0.615537},
 {37., 0.730724}, {37.5, 0.653848}, {38., 0.852311}, {38.5, 1.12861},
 {39., 1.75217}, {39.5, 2.80114}, {40., 4.90199}, {40.5, 3.57702}, {41., 2.20397},
 {41.5, 1.40104}, {42., 1.14951}, {42.5, 0.880876}, {43., 0.603862},
 {43.5, 0.463206}, {44., 0.486492}, {44.5, 0.426322}, {45., 0.674082}}
```

As you can see, we used a Lorentzian-type function to generate the data (Equation 11), along with some added random noise.

Next plot your data set to check it. You may be tempted to skip this step, but don't;
if you do not check your work as you go (in this case, your data entry), errors often creep in.
This plots the points:

In[166]:=
```
ListPlot[data, PlotRange → {{34, 46}, {0, 5.5}}]
```

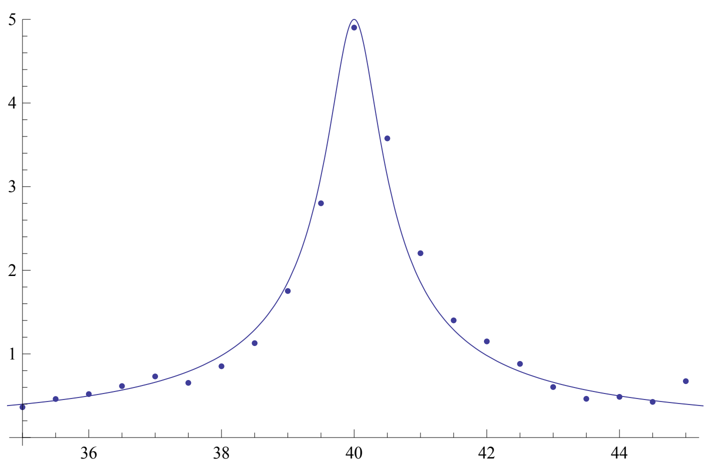
Out[166]=



Next you should try to "fit" your data by hand. To do this, guess a curve and plot it along with your data points.
Again, you may be tempted to skip this step and jump right to letting the computer fit your data.
But again, it is better to do this step first. You will need some initial guesses for the parameters when you do the nonlinear fit,
plus you just want to know what a good fit looks like. As a general rule, if you cannot see how to draw a line through your data, the computer will not know how to do it either.
Here is how to plot a fit line along with your data:

In[167]:=
```
amp = 5;  (* the semicolons suppresse the output for these lines *)
freq0 = 40; (* our guess for the center frequency *)
qq = 50; (* our guess for Q *)
Show[ListPlot[data],
 Plot[((amp * freq0) / (2 * qq)) / Sqrt[((x - freq0) ^2 + ((freq0 / (2 * qq)) ^2))],
  {x, 25, 55}, PlotRange → All]]
```

Out[170]=



To try a different curve, simply change a parameter and hit Enter again; the computer overwrites the plot. Note that Equation 11 was written in this form so the parameters are fairly orthogonal in their effect

-- **amp** changes the height of the peak, **freq0** changes the position of the peak, and **qq** changes the width of the peak.
Continue tweaking the parameters until you have a line that seems to go through your data reasonably well.
This eyeball "fit" is sometimes called "chi-by-eye".

Next use your fit parameters as input to help the nonlinear fit converge. (*Mathematica* is unhappy if the parameters in the model equal variables already used. So our parameters (amp,freq0,qq) become (amp1,freq01,qq1) in the fit:

In[171]:=
```
fit = NonlinearModelFit[data,
  ((amp1 * freq01) / (2 * qq1)) / Sqrt[((x - freq01)^2 + ((freq01 / (2 * qq1))^2))],
  {{amp1, 5}, {qq1, 50}, {freq01, 40}}, x]
```

Out[171]= FittedModel$\left[\frac{2.03921}{\sqrt{0.162067+(\ll 1 \gg)^2}}\right]$

This next line shows the best fit curve in a better format,
and it gives you the fit values of the parameters with standard errors.

In[172]:=
```
fit[{"BestFit", "ParameterTable"}]
```

Out[172]= $\{2.03921 / (\sqrt{(0.162067 + (-40.1095 + x)^2)})$,

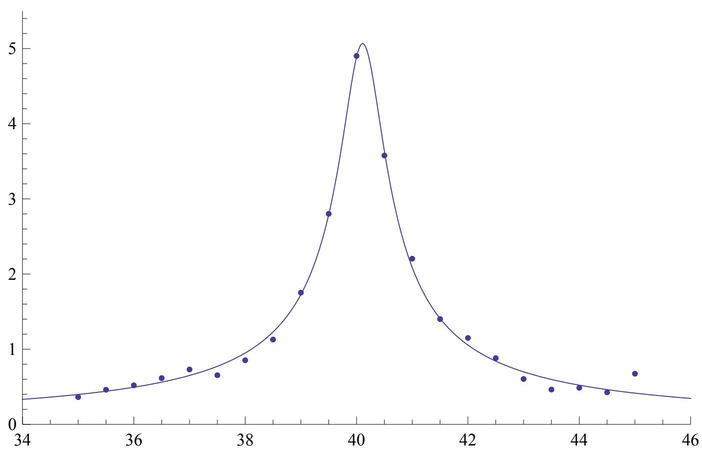
	Estimate	Standard Error	t-Statistic	P-Value
amp1	5.0654	0.108353	46.7491	3.01547×10^{-20}
qq1	49.8161	1.82297	27.3269	4.15566×10^{-16}
freq01	40.1095	0.0146292	2741.74	4.80001×10^{-52}

}

Note that *Mathematica* found a fit freq01 that is closer to the correct value of 40.1 Hz.

Finally we plot the data points with the fit curve ... not bad ....

In[184]:=
```
Show[ListPlot[data, PlotRange → {{34, 46}, {0, 5.5}}], Plot[fit[x], {x, 34, 46}]]
```

Out[184]=


Once you know the basics of Mathematica, you can expand your knowledge quickly using Google.

Search using “mathematica” along with whatever you are trying to do.

Fitting MMHO data, week 2: Oscillator frequency vs. Magnetic Field

For this measurement, you recorded the oscillator frequency as you applied different currents to the bias coils.
Again, start by entering your data into a *Mathematica* list like so (only showing two points here):

In[174]:=
```
data1 = {{current1, freq1}, {current2, freq2}}
```

Out[174]=
```
{{current1, freq1}, {current2, freq2}}
```

As above, we will use some simulated data:

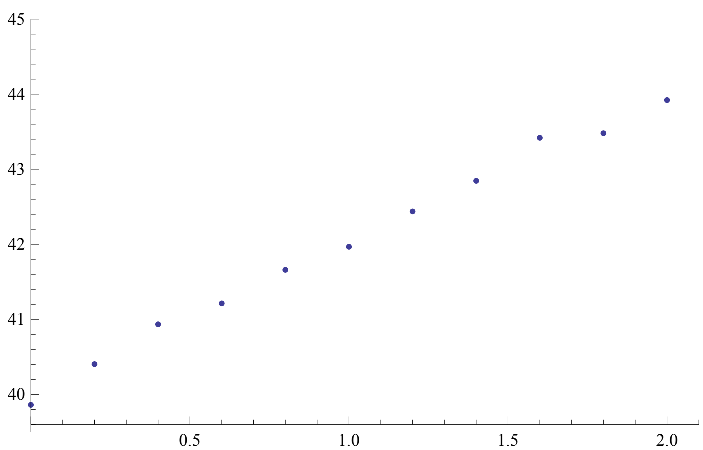
In[175]:=
```
freq0 = 40; (* the frequency with no applied current *)
dfreqdi = 2; (* the change with current *)
data2 = Table[
  {x, freq0 + dfreqdi * x + 0.1 * RandomVariate[NormalDistribution[]]}, {x, 0, 2, 0.2}]
```

Out[177]=
```
{{0., 39.8607}, {0.2, 40.4032}, {0.4, 40.934},
 {0.6, 41.2123}, {0.8, 41.6595}, {1., 41.9668}, {1.2, 42.4377},
 {1.4, 42.8451}, {1.6, 43.4186}, {1.8, 43.4795}, {2., 43.9208}}
```

And we plot the data to check it.

In[178]:=
```
ListPlot[data2, PlotRange → {{0, 2.1}, {39.5, 45}}]
```

Out[178]=



Then do a linear fit to the data (no parameter guesses are needed for linear fits; they pretty much always work as expected):

In[179]:=
```
fitline = LinearModelFit[data2, {1, x}, x]
```

Out[179]=
```
FittedModel[ 40.0079 + 2.0047 x ]
```

And print out the parameters with standard errors :

In[180]:=
```
fitline["ParameterTable"]
```

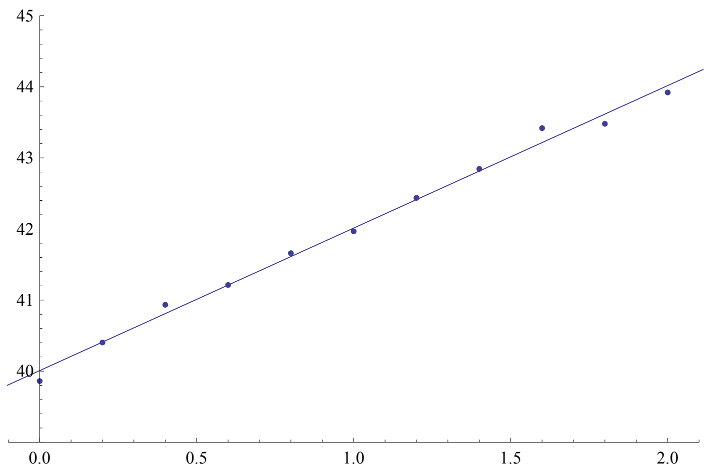
Out[180]=

	Estimate	Standard Error	t-Statistic	P-Value
1	40.0079	0.0630398	634.644	3.0483×10^{-22}
x	2.0047	0.0532784	37.6269	3.28188×10^{-11}

And finally plot the fit with the data points:

In[181]:=
```
Show[ListPlot[data2, PlotRange → {{-0.1, 2.1}, {39, 45}}],
 Plot[fitline[x], {x, -1, 3}]]
```

Out[181]=



And finally, you can use the fit slope to extract the magnetic moment of the test mass, as described in the handout.