Ph 22.3 — N-body Simulations

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Introduction

In this assignment we will explore the nontrivial behavior of physical systems consisting of a large number (N) of point masses (henceforth, "particles") subject to their reciprocal gravitational attraction. Such systems are representative of real physical systems such as *globular clusters*, which are aggregates of $\sim 10^5$ stars copiously present in many galaxies.

Large modern parallel computers can model these systems reasonably accurately. However, the computational cost of simulating their evolution rises very rapidly with N: in principle, it is necessary to compute the force exerted by each particle on every other particle, so the number of mathematical operations involved is N^2, which quickly becomes too much as N increases. Two approaches have been devised to deal with this limit. The first approach is to integrate the exact (*direct*) equations of motion on *special-purpose* computers in a brute-force fashion. This takes a lage number of computer cycles, but is reasonably straighforward. The second approach is to use approximated expressions that treat the force exerted by nearby particles exactly, but average the effects of more distant particles; this is the case, for instance, of the Barnes-Hut tree algorithm [J. Barnes and P. Hut, *Nature* **324**, 446 (1986)], where the number of mathematical operations scales as $N \log N$.

In this assignment we will use the brute-force to perform an N-body simulation. For the sake of mathematical simplicity, we will write a direct integrator, whereby all the N^2 forces are evaluated explicitly. The lab hardware is much slower than large parallel computers, so we will be able to simulate only a couple hundred or so particles with acceptable performance. However, even such small systems can exhibit interesting properties such as *core collapse* (the formation of a dense central core) and *halo formation* (the formation of a population of stars on far-out orbits, some of which are escaping).

General behaviour of the N-body problem

Consider the problem where we have a reasonably large number N point masses of total mass M localized within a region of characteristic size R, and the typical velocity of the masses in the centre of mass frame is v_0. What will happen?

The first thing to look at is the energetics. The total energy of the system is the sum of its kinetic energy E_{kin} and its gravitational potential energy E_{grav}:

$$E_{kin} = \sum_i \frac{1}{2} m_i v_i^2 \sim \frac{1}{2} M v_0^2 \tag{1}$$

$$E_{grav} = -\frac{1}{2} \sum_{i \neq j} \frac{G m_i m_j}{r_{ij}} \sim -\frac{GM^2}{R}. \tag{2}$$

If $E_{kin} > |E_{grav}|$, the total energy is positive and the system is unbound and will proceed to fly apart in short order. If $E_{kin} < |E_{grav}|$, some significant fraction of the masses will not have enough energy to escape the system, so a gravitationally-bound system will form, in which the masses will orbit under the combined gravitational pull of all other masses: this is what is happening in a galaxy or a star cluster.

A gravitationally-bound system in equilibrium is subject the *virial theorem*:

$$\langle E_{kin} \rangle = -\frac{1}{2} \langle E_{grav} \rangle, \tag{3}$$

where $\langle \cdot \rangle$ denotes the time average as $t \to \infty$. If a bound system does not initially satisfy the virial theorem, it will tend to rearrange its orbits until it does, eventually achieving some equilibrium. This process of relaxation toward equilibrium is known as *virialization*.

We can roll the range of possible behaviours into a single dimensionless parameter, known as the virial parameter:

$$\alpha \equiv 2E_{kin}/|E_{grav}| \sim \frac{v_0^2 R}{GM}. \tag{4}$$

If $\alpha >> 1$, the system will fly apart. If $\alpha \sim 1$, the system is "close" to virial equilibrium and will remain roughly the same size. If $\alpha << 1$, the system will collapse, converting gravitational energy to kinetic energy until $\alpha \sim 1$.

Timescales

On what timescale do we expect all the action to happen? By dimensional analysis, we can derive the timescale over which we expect gravity to be important, known as the *dynamical time*:

$$t_{dyn} = \sqrt{\frac{R^3}{GM}}. \tag{5}$$

Related timescales include the orbital time $t_{orb} = 2\pi t_{dyn}$, the time to complete a circular orbit around the periphery of the system, and the freefall time $t_{ff} = \frac{\pi}{2\sqrt{2}} t_{dyn}$, the time for a sphere of size R and mass M to collapse to a point. The initial virialization of the system will generally take a few t_{dyn}. During this time, the orbital energies of individual bodies will tend to undergo large fluctuations because the gravitational potential is changing greatly. By so rearranging the orbital energies the system can reach equilibrium. This phenomenon has the oxymoronic name *violent relaxation* [D. Lynden-Bell, *MNRAS*, **136** 101L (1967)], and is the principal mechanism by which galaxies reach a new equilibrium when they collide and merge with each other.

In general, to capture the dynamics of your system you should set your timestep to some small fraction of the dynamical time. Beware, however: if your system collapses, R will decrease and you will need yet shorter timesteps. It's always a good idea to vary the timestep to check for convergence.

Once virial equilibrium has been achieved, a different mechanism drives the evolution of the system. When two particles come close to each other on a chance encounter, they will deflect each other's trajectories and exchange some amount of energy and angular momentum. After many such encounters, a body can end up on a completely different orbit from the one it started on. The timescale for this to happen, the *2-body relaxation* timescale, is related to the dynamical time as [L. Spitzer & M. Hart, *ApJ*, **164** 399S (1971)]:

$$t_{relax} \sim \frac{N}{10 \ln N} t_{dyn}, \tag{6}$$

where N is the number of bodies in the system. For galaxies, which have upwards of 10^7 stars and $t_{dyn} \sim 100$Myr, t_{relax} is many times the age of the Universe, so 2-body relaxation can be neglected in galactic dynamics. However, for star clusters, with N ranging from 10^2 to 10^7 and $t_{dyn} < 10$Myr, t_{relax} can be significantly shorter than the age of the Universe, so a careful treatment of close stellar encounters is necessary to simulate them. In this assignment, with $N \sim 100$, t_{relax} is only a few dynamical times, so it is hard to isolate 2-body relaxation from violent relaxation.

The Assignment

1. Write a Python program that integrates Newton's equations of gravitation for a system of N particles moving *on a plane*, using a fixed-timestep *symplectic* Euler integrator (see Assignment Ph20.3 from Physics 20) or a higher-order symplectic integrator of your choice. Every few timesteps, visualize their positions (using graphical output).

   Because we can only afford to simulate a small number of particles, mass distribution will be much lumpier than in realistic systems. As a result, gravitational forces will tend to be more irregular, and the effect of close encounters between particles will be overemphasized. In practice, you will see that your clusters tend to break up rapidly. A partial fix to this problem is *force softening*: instead of Newton's expression, we use a gravitational force with the same direction, but proportional to $m_1 m_2/(r_{12}^2 + a^2)$, where r_{12} is the distance between m_1 and m_2, and where a is a constant.

2. Now set up a system of particles randomly distributed within a sphere of size one (in arbitrary units), with random initial velocities, and run the simulation. It is a bit hard to find the right parameters for a sensible simulation: for unit masses and unit G, I have had good results with velocities of all magnitude 0.1 (but in random directions), with $a = 0.1$, and with timestep between 0.01 and 0.001. Try also the very special initial condition where all the particles have zero velocity.

   A good way to monitor the evolution is to plot the distribution of radial positions and velocities once every few timesteps. For sufficiently large systems and sufficiently long integration times, you should be able to observe formation of a *core* and a *halo*. Show interesting movies to your TA.